4-4 Graphing Sine and Cosine Functions

Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.

1. $f(x) = \sin x$; $g(x) = \frac{1}{2} \sin x$

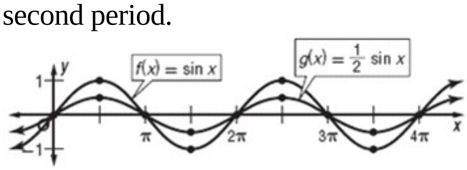
SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ compressed vertically. The amplitude of $g(x)$ is $\left|\frac{1}{2}\right|$ or $\frac{1}{2}$.

Create a table listing the coordinates of the x-intercepts and extrema for $f(x) = \sin x$ for one period, 2π, on the interval $[0, 2\pi]$. Then use the amplitude of $g(x)$ to find corresponding points on its graph.

Functions	$f(x) = \sin x$	$g(x) = \frac{1}{2}\sin x$
x-int	$(0, 0)$	$(0, 0)$
Max	$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{\pi}{2}, \frac{1}{2}\right)$
x-int	$(\pi, 0)$	$(\pi, 0)$
Min	$\left(\frac{3\pi}{2}, -1\right)$	$\left(\frac{3\pi}{2}, -\frac{1}{2}\right)$
x-int	$(2\pi, 0)$	$(2\pi, 0)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



2. $f(x) = \cos x; g(x) = -\frac{1}{3}\cos x$

SOLUTION:

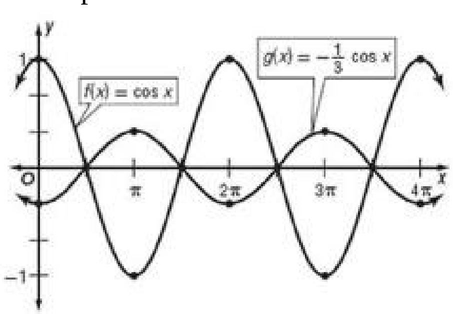
The graph of $g(x)$ is the graph of $f(x)$ compressed vertically. The amplitude of $g(x)$ is $\left|-\frac{1}{3}\right|$ or $\frac{1}{3}$.

Create a table listing the coordinates of the x-intercepts and extrema for $f(x) = \cos x$ for one period, 2π, on the interval $[0, 2\pi]$. Then use the amplitude of $g(x)$ to find corresponding points on its graph.

Function	$f(x) = \cos x$
Max	$(0, 1)$
x-int	$\left(\frac{\pi}{2}, 0\right)$
Min	$(\pi, -1)$
x-in	$\left(\frac{3\pi}{2}, 0\right)$
Min	$(2\pi, 1)$
x-int	$(2\pi, 0)$

Function	$f(x) = -\frac{1}{3}\cos x$
Max	$\left(0, -\frac{1}{3}\right)$
x-int	$\left(\frac{\pi}{2}, 0\right)$
Min	$\left(\pi, \frac{1}{3}\right)$
x-in	$\left(\frac{3\pi}{2}, 0\right)$
Min	$\left(2\pi, -\frac{1}{3}\right)$
x-int	$(2\pi, 0)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



4-4 Graphing Sine and Cosine Functions

3. $f(x) = \cos x$; $g(x) = 6 \cos x$

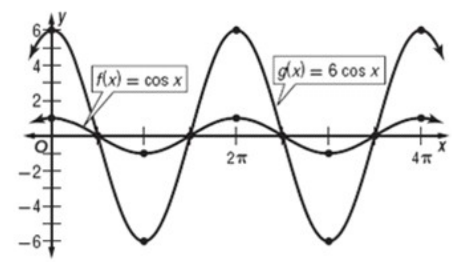
SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ expanded vertically. The amplitude of $g(x)$ is $|6|$ or 6.

Create a table listing the coordinates of the x-intercepts and extrema for $f(x) = \cos x$ for one period, 2π, on the interval $[0, 2\pi]$. Then use the amplitude of $g(x)$ to find corresponding points on its graph.

Functions	$f(x) = \cos x$	$g(x) = 6 \cos x$
Max	$(0, 1)$	$(0, 6)$
***x*-int**	$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{\pi}{2}, 0\right)$
Min	$(\pi, -1)$	$(\pi, -6)$
***x*-int**	$\left(\frac{3\pi}{2}, 0\right)$	$\left(\frac{3\pi}{2}, 0\right)$
Max	$(2\pi, 1)$	$(2\pi, 6)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



4. $f(x) = \sin x$; $g(x) = -8 \sin x$

SOLUTION:

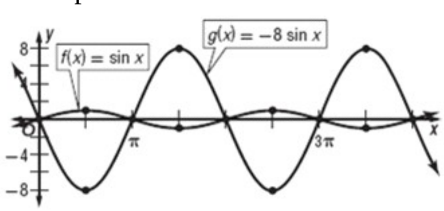
The graph of $g(x)$ is the graph of $f(x)$ expanded vertically. The amplitude of $g(x)$ is $|-8|$ or 8.

Create a table listing the coordinates of the x-intercepts and extrema for $f(x) = \sin x$ for one period, 2π, on the interval $[0, 2\pi]$. Then use the amplitude of $g(x)$ to find corresponding points on its graph.

Function	$f(x) = \sin x$
***x*-int**	$(0, 0)$
Max	$\left(\frac{\pi}{2}, 1\right)$
***x*-int**	$(\pi, 0)$
Min	$\left(\frac{3\pi}{2}, -1\right)$
***x*-int**	$(2\pi, 0)$

Function	$g(x) = -8 \sin x$
***x*-int**	$(0, 0)$
Min	$\left(\frac{\pi}{2}, -8\right)$
***x*-int**	$(\pi, 0)$
Max	$\left(\frac{3\pi}{2}, 8\right)$
***x*-int**	$(2\pi, 0)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



4-4 Graphing Sine and Cosine Functions

Describe how the graphs of $f(x)$ and $g(x)$ are related. Then find the period of $g(x)$, and sketch at least one period of both functions on the same coordinate axes.

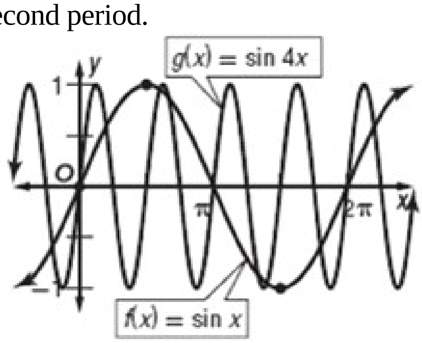
5. $f(x) = \sin x$; $g(x) = \sin 4x$

SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally. The period of $g(x)$ is $\frac{2\pi}{|4|} = \frac{\pi}{2}$. To find corresponding points on the graph of $g(x)$, change the x-coordinates of those key points on $f(x)$ so that they range from 0 to $\frac{\pi}{2}$, increasing by increments of $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.

Functions	$f(x) = \sin x$	$g(x) = \sin 4x$
x-int	(0, 0)	(0, 0)
Max	$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{\pi}{8}, 1\right)$
x-int	$(\pi, 0)$	$\left(\frac{\pi}{4}, 0\right)$
Min	$\left(\frac{3\pi}{2}, -1\right)$	$\left(\frac{3\pi}{8}, -1\right)$
x-int	$(2\pi, 0)$	$\left(\frac{\pi}{2}, 0\right)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



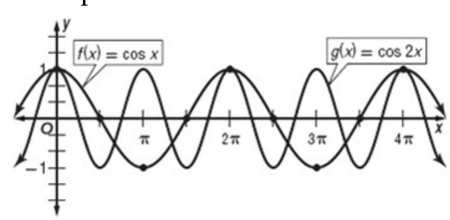
6. $f(x) = \cos x$; $g(x) = \cos 2x$

SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally. The period of $g(x)$ is $\frac{2\pi}{|2|}$ or π. To find corresponding points on the graph of $g(x)$, change the x-coordinates of those key points on $f(x)$ so that they range from 0 to π, increasing by increments of $\frac{\pi}{4}$.

Functions	$f(x) = \cos x$	$g(x) = \cos 2x$
Max	(0, 1)	(0, 1)
x-int	$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{\pi}{4}, 0\right)$
Min	$(\pi, -1)$	$\left(\frac{\pi}{2}, -1\right)$
x-int	$\left(\frac{3\pi}{2}, 0\right)$	$\left(\frac{3\pi}{4}, 0\right)$
Max	$(2\pi, 1)$	$(\pi, 1)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



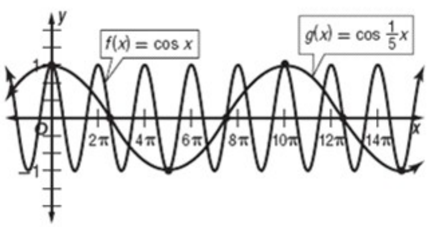
7. $f(x) = \cos x$; $g(x) = \cos \frac{1}{5}x$

SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally. The period of $g(x)$ is $\frac{2\pi}{\left|\frac{1}{5}\right|} = 10\pi$. To find corresponding points on the graph of $g(x)$, change the x-coordinates of those key points on $f(x)$ so that they range from 0 to 10π, increasing by increments of $\frac{10\pi}{4}$ or $\frac{5\pi}{2}$.

Functions	$f(x) = \cos x$	$g(x) = \cos \frac{1}{5}x$
Max	(0, 1)	(0, 1)
***x*-int**	$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{5\pi}{2}, 0\right)$
Min	$(\pi, -1)$	$(5\pi, -1)$
***x*-int**	$\left(\frac{3\pi}{2}, 0\right)$	$\left(\frac{15\pi}{2}, 0\right)$
Max	$(2\pi, 1)$	$(10\pi, 1)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



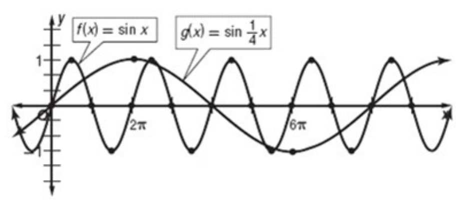
8. $f(x) = \sin x$; $g(x) = \sin \frac{1}{4}x$

SOLUTION:

The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally. The period of $g(x)$ is $\frac{2\pi}{\left|\frac{1}{4}\right|} = 8\pi$. To find corresponding points on the graph of $g(x)$, change the x-coordinates of those key points on $f(x)$ so that they range from 0 to 8π, increasing by increments of $\frac{8\pi}{4}$ or 2π.

Functions	$f(x) = \sin x$	$g(x) = \sin \frac{1}{4}x$
***x*-int**	(0, 0)	(0, 0)
Max	$\left(\frac{\pi}{2}, 1\right)$	$(2\pi, 1)$
***x*-int**	$(\pi, 0)$	$(4\pi, 0)$
Min	$\left(\frac{3\pi}{2}, -1\right)$	$(6\pi, -1)$
***x*-int**	$(2\pi, 0)$	$(8\pi, 0)$

Sketch the curve through the indicated points for each function. Then repeat the pattern to complete a second period.



9. **VOICES** The contralto vocal type includes the deepest female singing voice. Some contraltos can sing as low as the E below middle C (E3), which has a frequency of 165 hertz. Write an equation for a sine function that models the initial behavior of the sound wave associated with E3 having an amplitude of 0.15.

SOLUTION:

The general form of the equation will be $y = a \sin bt$, where t is the time in seconds. Because the amplitude is 0.15, $|a| = 0.15$. This means that $a = \pm 0.15$.

The period is the reciprocal of the frequency or $\frac{1}{165}$. Use this value to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{1}{165} = \frac{2\pi}{|b|}$$
$$\frac{|b|}{165} = 2\pi$$
$$|b| = 2\pi(165) \text{ or } 330\pi$$
$$b = \pm 330\pi$$

Sample answer: One sine function that models the initial behavior is $y = 0.15 \sin 330\pi t$.

Write a sine function that can be used to model the initial behavior of a sound wave with the frequency and amplitude given.

10. $f = 440$, $a = 0.3$

SOLUTION:

The general form of the equation is $y = a \sin bt$, where t is the time in seconds. Because the amplitude is 0.3, $|a| = 0.3$. This means that $a = \pm 0.3$.

The period is the reciprocal of the frequency or $\frac{1}{440}$. Use this value to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{1}{440} = \frac{2\pi}{|b|}$$
$$\frac{|b|}{440} = 2\pi$$
$$|b| = 2\pi(440) \text{ or } 880\pi$$
$$b = \pm 880\pi$$

Using the positive values of a and b, one sine function that models the initial behavior is $y = 0.3 \sin 880\pi t$.

11. $f = 932$, $a = 0.25$

SOLUTION:

The general form of the equation is $y = a \sin bt$, where t is the time in seconds. Because the amplitude is 0.25, $|a| = 0.25$. This means that $a = \pm 0.25$.

The period is the reciprocal of the frequency or $\frac{1}{932}$. Use this value to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{1}{932} = \frac{2\pi}{|b|}$$
$$\frac{|b|}{932} = 2\pi$$
$$|b| = 2\pi(932) \text{ or } 1864\pi$$
$$b = \pm 1864\pi$$

Using the positive values of a and b, one sine function that models the initial behavior is $y = 0.25 \sin 1864\pi t$.

12. $f = 1245$, $a = 0.12$

SOLUTION:

The general form of the equation is $y = a \sin bt$, where t is the time in seconds. Because the amplitude is 0.12, $|a| = 0.12$. This means that $a = \pm 0.12$.

The period is the reciprocal of the frequency or $\frac{1}{1245}$. Use this value to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{1}{1245} = \frac{2\pi}{|b|}$$
$$\frac{|b|}{1245} = 2\pi$$
$$|b| = 2\pi(1245) \text{ or } 2490\pi$$
$$b = \pm 2490\pi$$

Using the positive values of a and b, one sine function that models the initial behavior is $y = 0.12 \sin 2490\pi t$.

13. $f = 623$, $a = 0.2$

SOLUTION:

The general form of the equation is $y = a \sin bt$, where t is the time in seconds. Because the amplitude is 0.2, $|a| = 0.2$. This means that $a = \pm 0.2$.

The period is the reciprocal of the frequency or $\frac{1}{623}$. Use this value to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{1}{623} = \frac{2\pi}{|b|}$$
$$\frac{|b|}{623} = 2\pi$$
$$|b| = 2\pi(623) \text{ or } 1246\pi$$
$$b = \pm 1246\pi$$

Using the positive values of a and b, one sine function that models the initial behavior is $y = 0.2 \sin 1246\pi t$.

4-4 Graphing Sine and Cosine Functions

State the amplitude, period, frequency, phase shift, and vertical shift of each function. Then graph two periods of the function.

14. $y = 3\sin\left(x - \frac{\pi}{4}\right)$

SOLUTION:

In this function, $a = 3$, $b = 1$, $c = -\frac{\pi}{4}$, and $d = 0$.
Because $d = 0$, there is no vertical shift.

Amplitude: $|a| = |3|$ or 3

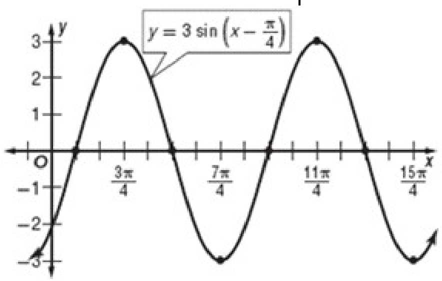
Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or 2π

Frequency: $\frac{|b|}{2\pi} = \frac{|1|}{2\pi}$ or $\frac{1}{2\pi}$

Phase shift: $-\frac{c}{|b|} = -\frac{\frac{\pi}{4}}{|1|}$ or $-\frac{\pi}{4}$

Midline: $y = d$ or $y = 0$

Graph $y = 3 \sin x$ shifted $\frac{\pi}{4}$ units to the right.



15. $y = \cos\left(\frac{x}{3} + \frac{\pi}{2}\right)$

SOLUTION:

In this function, $a = 1$, $b = \frac{1}{3}$, $c = \frac{\pi}{2}$, and $d = 0$.
Because $d = 0$, there is no vertical shift.

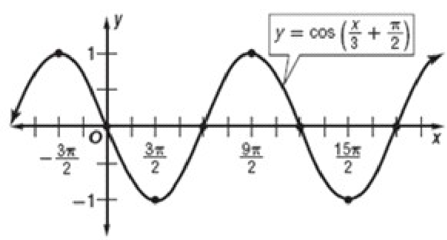
Amplitude: $|a| = |1|$ or 1

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{3}\right|}$ or 6π

Frequency: $\frac{|b|}{2\pi} = \frac{\left|\frac{1}{3}\right|}{2\pi}$ or $\frac{1}{6\pi}$

Phase shift: $-\frac{c}{|b|} = -\frac{\frac{\pi}{2}}{\left|\frac{1}{3}\right|}$ or $-\frac{3\pi}{2}$

Midline: $y = d$ or $y = 0$



4-4 Graphing Sine and Cosine Functions

16. $y = 0.25 \cos x + 3$

SOLUTION:

In this function, $a = \frac{1}{4}$, $b = 1$, $c = 0$, and $d = 3$.

$$\text{Amplitude: } |a| = \left|\frac{1}{4}\right| \text{ or } \frac{1}{4}$$

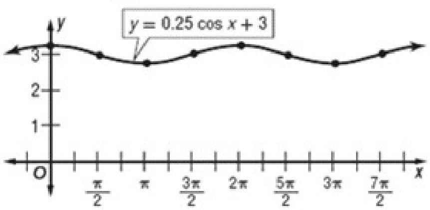
$$\text{Period: } \frac{2\pi}{|b|} = \frac{2\pi}{|1|} \text{ or } 2\pi$$

$$\text{Frequency: } \frac{|b|}{2\pi} = \frac{|1|}{2\pi} \text{ or } \frac{1}{2\pi}$$

$$\text{Phase shift: } -\frac{c}{|b|} = -\frac{0}{|1|} \text{ or } 0$$

$$\text{Midline: } y = d \text{ or } y = 3$$

Graph $y = \frac{1}{4} \cos x$ shifted 3 units up.



17. $y = \sin 3x - 2$

SOLUTION:

In this function, $a = 1$, $b = 3$, $c = 0$, and $d = -2$.

$$\text{Amplitude: } |a| = |1| \text{ or } 1$$

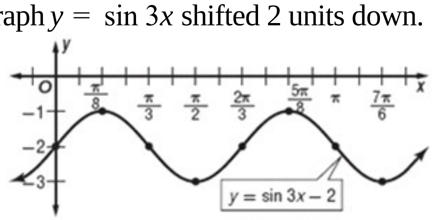
$$\text{Period: } \frac{2\pi}{|b|} = \frac{2\pi}{|3|} \text{ or } \frac{2\pi}{3}$$

$$\text{Frequency: } \frac{|b|}{2\pi} = \frac{|3|}{2\pi} \text{ or } \frac{3}{2\pi}$$

$$\text{Phase shift: } -\frac{c}{|b|} = -\frac{0}{|3|} \text{ or } 0$$

$$\text{Midline: } y = d \text{ or } y = -2$$

Graph $y = \sin 3x$ shifted 2 units down.



18. $y = \cos\left(x - \frac{3\pi}{2}\right) - 1$

SOLUTION:

In this function, $a = 1$, $b = 1$, $c = -\frac{3\pi}{2}$, and $d = -1$.

.

$$\text{Amplitude: } |a| = |1| \text{ or } 1$$

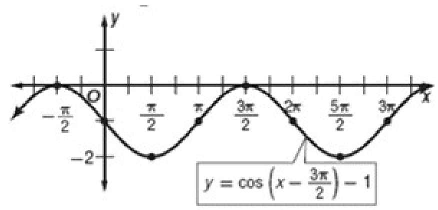
$$\text{Period: } \frac{2\pi}{|b|} = \frac{2\pi}{|1|} \text{ or } 2\pi$$

$$\text{Frequency: } \frac{|b|}{2\pi} = \frac{|1|}{2\pi} \text{ or } \frac{1}{2\pi}$$

$$\text{Phase shift: } -\frac{c}{|b|} = -\frac{-\frac{3\pi}{2}}{|1|} \text{ or } \frac{3\pi}{2}$$

$$\text{Midline: } y = d \text{ or } y = -1$$

Graph $y = \cos x$ shifted $\frac{3\pi}{2}$ units to the right and 1 unit down.



19. $y = \sin\left(x + \frac{5\pi}{6}\right) + 4$

SOLUTION:

In this function, $a = 1$, $b = 1$, $c = \frac{5\pi}{6}$, and $d = 4$.

.

Amplitude: $|a| = |1|$ or 1

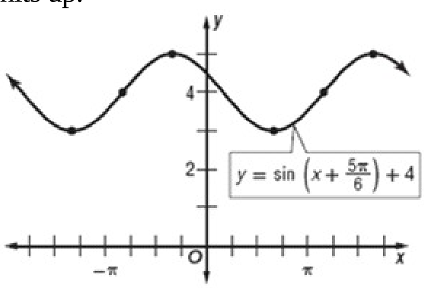
Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or 2π

Frequency: $\frac{|b|}{2\pi} = \frac{|1|}{2\pi}$ or $\frac{1}{2\pi}$

Phase shift: $-\frac{c}{|b|} = -\frac{\frac{5\pi}{6}}{|1|}$ or $-\frac{5\pi}{6}$

Midline: $y = d$ or $y = 4$

Graph $y = \sin x$ shifted $\frac{5\pi}{6}$ units to the left and 4 units up.



20. **VACATIONS** The average number of reservations R that a vacation resort has at the beginning of each month is shown.

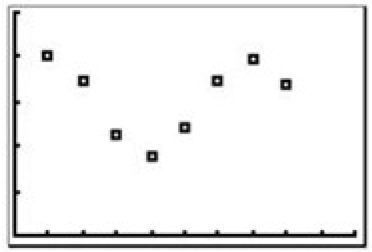
Month	R	Month	R
Jan	200	May	121
Feb	173	Jun	175
Mar	113	Jul	198
Apr	87	Aug	168

a. Write an equation of a sinusoidal function that models the average number of reservations using x = 1 to represent January.

b. According to your model, approximately how many reservations can the resort anticipate in November?

SOLUTION:

a. Make a scatter plot of the data and choose a model.



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Sample answer: Use a sinusoidal function of the form $y = a \cos(bx + c) + d$ to model the data. First, find the maximum M and minimum m values of the data, and use these values to find a, b, c, and d.

The maximum and minimum reservations are 200 and 87, respectively. The amplitude a is half of the distance between these values.

$$a = \frac{1}{2}(M - m)$$
$$= \frac{1}{2}(200 - 87) \text{ or } 56.5$$

The vertical shift d is the average of the maximum and minimum data values.

$$d = \frac{1}{2}(M + m)$$
$$= \frac{1}{2}(200 + 87) \text{ or } 143.5$$

There are 3 months between the maximum and minimum, so the period is 6.

$$|b| = \frac{2\pi}{\text{period}}$$
$$|b| = \frac{2\pi}{6}$$
$$|b| = \frac{\pi}{3}$$

The maximum data value occurs when $x = 1$. Since $y = \cos x$ attains its first maximum when $x = 0$, we must apply a phase shift of 1 unit. Use this value to find c.

$$\text{Phase shift} = -\frac{c}{|b|}$$
$$1 = -\frac{c}{\frac{\pi}{3}}$$
$$1 = -\frac{3c}{\pi}$$
$$\pi = -3c$$
$$-\frac{\pi}{3} = c$$

Write a function using the values for a, b, c, and d. Use $b = \frac{\pi}{3}$.

$$y = 56.5 \cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 143.5$$

b. To find the number of reservations in November, evaluate the model for $x = 11$.

$$y = 56.5 \cos\left(\frac{\pi}{3}(11) - \frac{\pi}{3}\right) + 143.5$$
$$\approx 115$$

Therefore, the resort can anticipate about 115 reservations in November.

21. **TIDES** The table shown below provides data for the first high and low tides of the day for a certain bay during one day in June.

Tide	Height (ft)	Time
first high tide	12.95	4:25 A.M.
first low tide	2.02	10:55 A.M.

a. Determine the amplitude, period, phase shift, and vertical shift of a sinusoidal function that models the height of the tide. Let x represent the number of hours that the high or low tide occurred after midnight.
b. Write a sinusoidal function that models the data.
c. According to your model, what was the height of the tide at 8:45 P.M. that night?

SOLUTION:

a. Sample answer: Use a sinusoidal function of the form $y = a \cos (bx + c) + d$ to model the data. First, find the amplitude a, which is half of the distance between the heights of the high and low tides.

$$a = \frac{1}{2}(M - m)$$
$$= \frac{1}{2}(12.95 - 2.02) \text{ or } 5.465$$

The vertical shift d is the average of the heights of the high and low tides.

$$d = \frac{1}{2}(M + m)$$
$$= \frac{1}{2}(12.95 + 2.02) \text{ or } 7.485$$

A sinusoid completes half of a period in the time that it takes to go from its maximum to its minimum value. One period is twice this time. The first high tide occurs 6.5 hours before the first low tide, so the period is 13.

$$|b| = \frac{2\pi}{\text{period}} = \frac{2\pi}{13} = \frac{\pi}{6.5}$$

The maximum data value occurs when $x \approx 4.417$. Since $y = \cos x$ attains its first maximum when $x = 0$, we must apply a phase shift of 4.417 units. Use this value to find c.

$$\text{Phase shift} = -\frac{c}{|b|}$$
$$4.417 = -\frac{c}{\frac{\pi}{6.5}}$$
$$4.417 = -\frac{6.5c}{\pi}$$
$$4.417\pi = -6.5c$$
$$-\frac{\pi}{1.47} = c$$

b. Write a function using the values for a, b, c, and d. Use $b = \frac{\pi}{6.5}$.

$$y = 5.465\cos\left(\frac{\pi}{6.5}x - \frac{\pi}{1.47}\right) + 7.485$$

c. To find the height of the water at 8:45 P.M. that night, evaluate the model for $x = 20.75$.

$$y = 5.465\cos\left(\frac{\pi}{6.5}x - \frac{\pi}{1.47}\right) + 7.485$$
$$\approx 5.465\cos\left(\frac{\pi}{6.5}(20.75) - \frac{\pi}{1.47}\right) + 7.485$$
$$\approx 7.28$$

Therefore, the tide was about 7.28 feet high at 8:45 P.M. that night.

22. **METEOROLOGY** The average monthly temperatures for Boston, Massachusetts are shown.

Month	Temp. (°F)	Month	Temp. (°F)
Jan	29	Jul	74
Feb	30	Aug	72
Mar	39	Sept	65
Apr	48	Oct	55
May	58	Nov	45
Jun	68	Dec	34

a. Determine the amplitude, period, phase shift, and vertical shift of a sinusoidal function that models the monthly temperatures using $x = 1$ to represent January.

b. Write an equation of a sinusoidal function that models the monthly temperatures.

c. According to your model, what is Boston's average temperature in August?

SOLUTION:

a. The maximum and minimum are 74° and 29°, respectively. The amplitude a is half of the distance between these values.

$$a = \frac{1}{2}(M - m)$$
$$= \frac{1}{2}(74 - 29) \text{ or } 22.5$$

The vertical shift d is the average of the maximum and minimum data values.

$$d = \frac{1}{2}(M + m)$$
$$= \frac{1}{2}(74 + 29) \text{ or } 51.5$$

A sinusoid completes half of a period in the time that it takes to go from its maximum to its minimum value. So, one period is twice this time. The maximum function value occurs in July, so x_{max} is 7. The minimum function value occurs in January, so x_{min} is 1. The period is 2(6) = 12.

$$|b| = \frac{2\pi}{12} \text{ or } \frac{\pi}{6}$$

The maximum data value occurs when $x = 7$. Since $y = \cos x$ attains its first maximum when $x = 0$, we must apply a phase shift of 7 units. Use this value to find c.

$$\text{Phase shift} = -\frac{c}{|b|}$$
$$7 = -\frac{c}{\frac{\pi}{6}}$$
$$7 = -\frac{6c}{\pi}$$
$$7\pi = -6c$$
$$-\frac{7\pi}{6} = c$$

b. Using the values for a, b, c, and d from part **a**, a sinusoidal function that models that monthly temperatures in Boston is

$$y = 22.5 \cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 51.5.$$

c. To find the average monthly temperature for Boston in August, evaluate the model for $x = 8$.

$$y = 22.5 \cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 51.5$$
$$= 22.5 \cos\left(\frac{\pi}{6}(8) - \frac{7\pi}{6}\right) + 51.5$$
$$\approx 70.99$$

Therefore, the average monthly temperature for Boston in August is about 71°F.

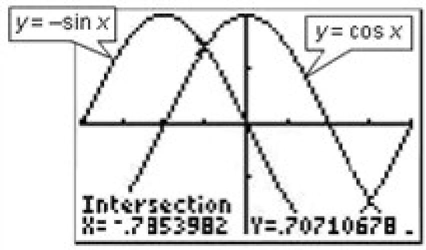
4-4 Graphing Sine and Cosine Functions

GRAPHING CALCULATOR Find the values of x in the interval $-\pi < x < \pi$ that make each equation or inequality true. (*Hint*: Use the intersection function.)

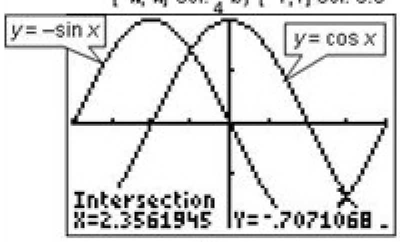
23. $-\sin x = \cos x$

SOLUTION:

Graph $y = -\sin x$ and $y = \cos x$ on the same graphing calculator screen on $(-\pi, \pi)$. Use the **intersect** feature under the **CALC** menu to determine where the two graphs intersect.



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1,1]$ scl: 0.5



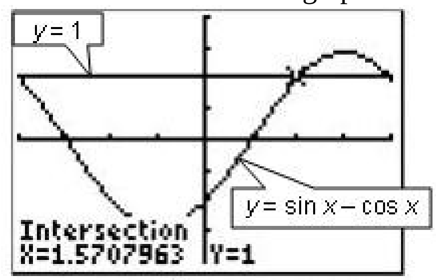
$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1,1]$ scl: 0.5

The graphs intersect at about -0.7854 or $-\frac{\pi}{4}$ and about 2.356 or $\frac{3\pi}{4}$. Therefore, on $-\pi < x < \pi$, $-\sin x = \cos x$ when $x = -\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

24. $\sin x - \cos x = 1$

SOLUTION:

Graph $y = \sin x - \cos x$ and $y = 1$ on the same graphing calculator screen on $(-\pi, \pi)$. Use the **intersect** feature under the **CALC** menu to determine where the two graphs intersect.



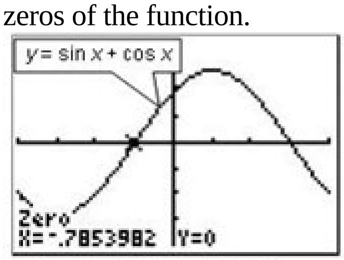
$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-2,2]$ scl: 0.5

The graphs intersect at about 1.57 or $\frac{\pi}{2}$. Therefore, on $-\pi < x < \pi$, $\sin x - \cos x = 1$ when $x = \frac{\pi}{2}$.

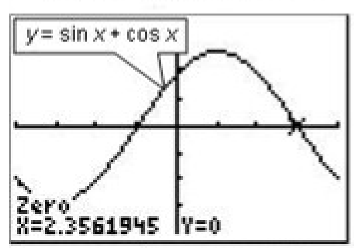
25. $\sin x + \cos x = 0$

SOLUTION:

Graph $y = \sin x + \cos x$ on $(-\pi, \pi)$. Use the **zero** feature under the **CALC** menu to determine the zeros of the function.



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-2,2]$ scl: 0.5



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-2,2]$ scl: 0.5

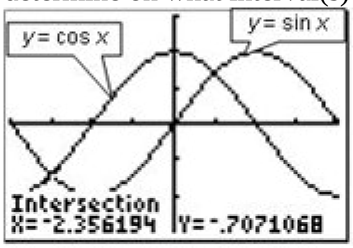
There are zeros at about -0.785 or $-\frac{\pi}{4}$ or and about 2.356 or $\frac{3\pi}{4}$. Therefore, on $-\pi < x < \pi$, $\sin x + \cos x = 0$ when $x = -\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

4-4 Graphing Sine and Cosine Functions

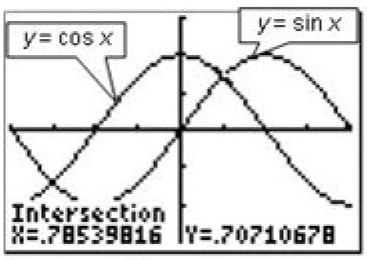
26. $\cos x \le \sin x$

SOLUTION:

Graph $y = \cos x$ and $y = \sin x$ on $(-\pi, \pi)$. Use the **intersect** feature under the **CALC** menu to determine on what interval(s) $\cos x \le \sin x$.



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1.5, 1.5]$ scl: 0.5



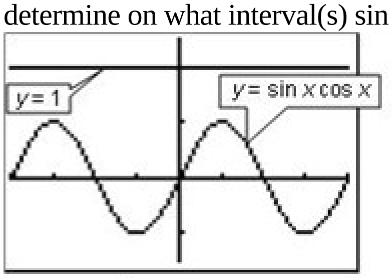
$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1.5, 1.5]$ scl: 0.5

The graphs intersect at about –2.356 or $-\frac{3\pi}{4}$ and about 0.785 or $\frac{\pi}{4}$. From the graph, it appears that $\cos x \le \sin x$ on $-\pi < x < -\frac{3\pi}{4}$ and $\frac{\pi}{4} < x < \pi$.

27. $\sin x \cos x > 1$

SOLUTION:

Graph $y = \sin x \cos x$ and $y = 1$ on $(-\pi, \pi)$. Use the **intersect** feature under the **CALC** menu to determine on what interval(s) $\sin x \cos x > 1$.



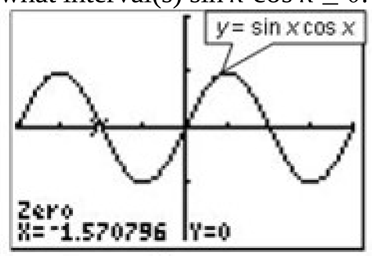
$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-0.75, 1.25]$ scl: 0.5

The graphs do not intersect. Therefore, $y = \sin x \cos x$ is not greater than $y = 1$ for any values of x on $-\pi < x < \pi$.

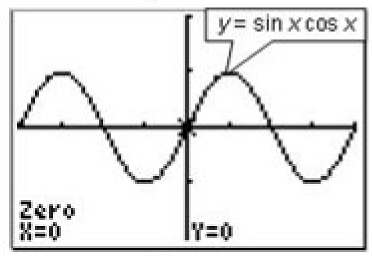
28. $\sin x \cos x \le 0$

SOLUTION:

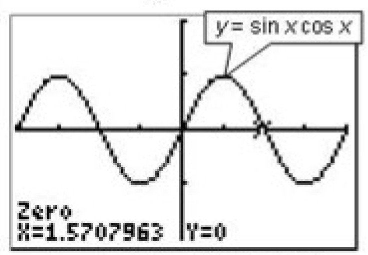
Graph $y = \sin x \cos x$ on $(-\pi, \pi)$. Use the **zero** feature under the **CALC** menu to determine on what interval(s) $\sin x \cos x \le 0$.



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1, 1]$ scl: 0.5



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1, 1]$ scl: 0.5



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1, 1]$ scl: 0.5

The zeros of $y = \sin x \cos x$ are about –1.57 or $-\frac{\pi}{2}$, 0, and about 1.57 or $\frac{\pi}{2}$. From the graph, it appears that $\sin x \cos x \le 0$ on $-\frac{\pi}{2} < x < 0$ and $\frac{\pi}{2} < x < \pi$.

29. **CAROUSELS** A wooden horse on a carousel moves up and down as the carousel spins. When the ride ends, the horse usually stops in a vertical position different from where it started. The position y of the horse after t seconds can be modeled by $y = 1.5 \sin (2t + c)$, where the phase shift c must be continuously adjusted to compensate for the different starting positions. If during one ride the horse reached a maximum height after $\frac{7\pi}{12}$ seconds, find the equation that models the horse's position.

SOLUTION:

First, find the period of $y = 1.5 \sin (2t + c)$.

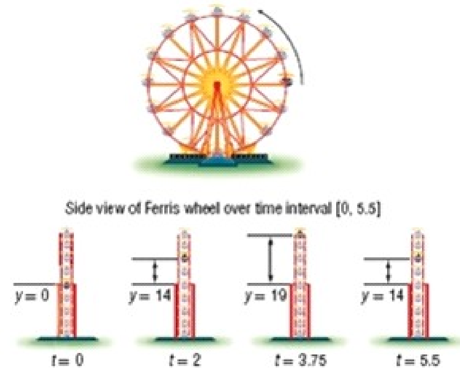
$$\text{period} = \frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \pi$$

Since the period is π, the function will reach a maximum height at $\frac{\pi}{4}$ radians. The phase shift is the difference between the horizontal position of the function at $\frac{\pi}{4}$ and $\frac{7\pi}{12}$ radians, which is $\frac{\pi}{3}$ radians. Substitute $\frac{\pi}{3}$ and b into the phase shift formula to find c.

$$\text{phase shift} = -\frac{c}{|b|}$$
$$\frac{\pi}{3} = -\frac{c}{|2|}$$
$$\frac{\pi}{3} = -\frac{c}{2}$$
$$2\left(\frac{\pi}{3}\right) = -c$$
$$-\frac{2\pi}{3} = c$$

Therefore, the equation is $y = 1.5\sin\left(2t - \frac{2\pi}{3}\right)$.

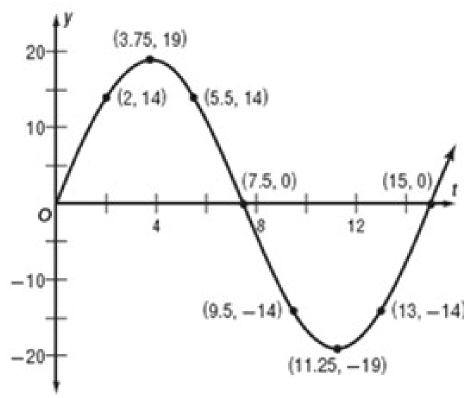
30. **AMUSEMENT PARKS** The position y in feet of a passenger cart relative to the center of a Ferris wheel over t seconds is shown below.



a. Find the time t that it takes for the cart to return to $y = 0$ during its initial spin.
b. Find the period of the Ferris wheel.
c. Sketch the graph representing the position of the passenger cart over one period.
d. Write a sinusoidal function that models the position of the passenger cart as a function of time t.

SOLUTION:

a. From the diagram, the height of the cart is 0 ft when $t = 0$, and reaches a maximum height of 19 feet when $t = 3.75$ seconds. So, the cart will return to $y = 0$ after 2(3.75) or 7.5 seconds.
b. The cart completes half of a cycle from $t = 0$ to $t = 7.5$. So, the period is 2(7.5) or 15 seconds.
c.



d. The amplitude a is half of the maximum and minimum values. So, $a = \frac{1}{2}[19 - (-19)]$ or 19.

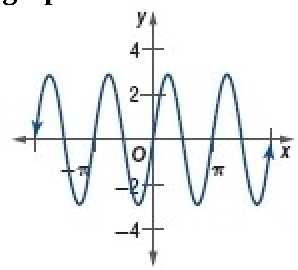
The vertical shift d is the average of the maximum and minimum data values. So, $d = \frac{1}{2}(19 - 19)$ or 0.

Because the period of a sinusoidal function is $\frac{2\pi}{|b|}$, you can write $|b| = \frac{2\pi}{\text{period}}$. Therefore, $|b| = \frac{2\pi}{15}$.

There is no phase shift because the height of the cart when $t = 0$ is 0 feet, and the sine function has a y-intercept of 0.

Therefore, one equation that could be used to model the position of the passenger cart is $y = 19 \sin \frac{2\pi}{15}t$.

Write an equation that corresponds to each graph.

31. 

SOLUTION:

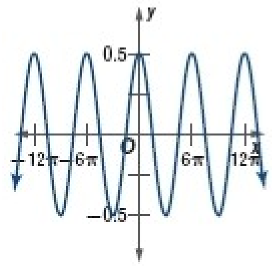
Sample answer: There is an x-intercept at 0, so one equation that corresponds to this graph is $y = a \sin (bx + c) + d$.

Half of the distance from the maximum to the minimum value of the function is 3. So, the amplitude is 3.

It appears that the function completes one period on $[0, \pi]$. Find b.

$$\begin{aligned} \text{period} &= \frac{2\pi}{|b|} \\ \pi &= \frac{2\pi}{|b|} \\ |b|\pi &= 2\pi \\ |b| &= 2 \\ b &= \pm 2 \end{aligned}$$

The midline appears to be at $y = 0$, so there does not appear to be any vertical shift. Therefore, one equation that corresponds to this graph is $y = 3 \sin (2x)$.

32. 

SOLUTION:

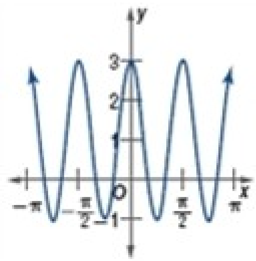
Sample answer: Because there is a y-intercept at $\frac{1}{2}$, one equation that corresponds to this graph is $y = a \cos (bx + c) + d$.

Half of the distance from the maximum to the minimum value of the function is $\frac{1}{2}$. So, the amplitude is $\frac{1}{2}$.

It appears that the function completes one period on $[0, 6\pi]$. Find b.

$$\begin{aligned} \text{period} &= \frac{2\pi}{|b|} \\ 6\pi &= \frac{2\pi}{|b|} \\ |b| 6\pi &= 2\pi \\ |b| &= \frac{1}{3} \\ b &= \pm \frac{1}{3} \end{aligned}$$

The midline appears to be at $y = 0$, so there does not appear to be any vertical shift. Therefore, one equation that corresponds to this graph is $y = \frac{1}{2} \cos \frac{x}{3}$.

33. 

SOLUTION:

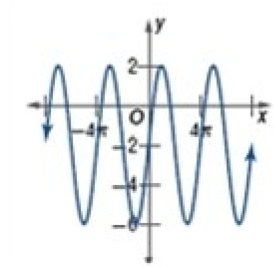
Sample answer: Because there is a y-intercept at 3, one equation that corresponds to this graph is $y = a \cos(bx + c) + d$.

Half of the distance from the maximum to the minimum value of the function is 4 ÷ 2 or 2. So, the amplitude is 2.

It appears that the function completes one period on $\left[0, \frac{\pi}{2}\right]$. Find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\frac{\pi}{2} = \frac{2\pi}{|b|}$$
$$|b|\frac{\pi}{2} = 2\pi$$
$$|b| = 4$$
$$b = \pm 4$$

The midline appears to be at $y = 1$, so there is a vertical shift 1 unit up. Therefore, one equation that corresponds to this graph is $y = 2 \cos 4x + 1$.

34. 

SOLUTION:

Sample answer: There is an x-intercept at 0, so one equation that corresponds to this graph is $y = a \sin(bx + c) + d$.

Half of the distance from the maximum to the minimum value of the function is 8 ÷ 2 or 4. So, the amplitude is 4.

It appears that the function completes one period on $[0, 4\pi]$. Find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$4\pi = \frac{2\pi}{|b|}$$
$$|b|4\pi = 2\pi$$
$$|b| = \frac{1}{2}$$
$$b = \pm\frac{1}{2}$$

The midline appears to be at $y = -2$, so there is a vertical shift of 2 units down. Therefore, one equation that corresponds to this graph is $y = 4 \sin \frac{x}{2} - 2$.

Write a sinusoidal function with the given period and amplitude that passes through the given point.

35. period: π; amplitude: 5; point: $\left(\frac{\pi}{6}, \frac{5}{2}\right)$

SOLUTION:

Use the period to find b.

$$\pi = \frac{2\pi}{|b|}$$
$$|b|\pi = 2\pi$$
$$|b| = 2$$
$$b = \pm 2$$

Sample answer: One sinusoidal function in which a = 5 and b = 2 is y = 5 cos 2x. Evaluate the function for $x = \frac{\pi}{6}$.

$$y = 5\cos 2x$$
$$y = 5\cos\left(2 \cdot \frac{\pi}{6}\right)$$
$$y = 5\cos\frac{\pi}{3}$$
$$x = 5\left(\frac{1}{2}\right) \text{ or } \frac{5}{2}$$

The function passes through $\left(\frac{\pi}{6}, \frac{5}{2}\right)$. Therefore, a sinusoidal function with period π and amplitude 5 that passes through the point $\left(\frac{\pi}{6}, \frac{5}{2}\right)$ is y = 5 cos 2x.

36. period: 4π; amplitude: 2; point: (π, 2)

SOLUTION:

Use the period to find b.

$$4\pi = \frac{2\pi}{|b|}$$
$$|b|4\pi = 2\pi$$
$$|b| = \frac{1}{2}$$
$$b = \pm\frac{1}{2}$$

Sample answer: One sinusoidal function in which a = 2 and $b = \frac{1}{2}$ is y = 2 sin $\frac{x}{2}$. Evaluate the function for $x = \pi$.

$$y = 2\sin\frac{x}{2}$$
$$y = 2\sin\frac{\pi}{2}$$
$$y = 2(1) \text{ or } 2$$

The function passes through (π, 2). Therefore, a sinusoidal function with period 4π and amplitude 2 that passes through the point (π, 2) is y = 2 sin $\frac{x}{2}$.

4-4 Graphing Sine and Cosine Functions

37. period: $\frac{\pi}{2}$; amplitude: $\frac{3}{2}$; point: $\left(\frac{\pi}{2}, \frac{3}{2}\right)$

SOLUTION:

Use the period to find b.

$$\frac{\pi}{2} = \frac{2\pi}{|b|}$$
$$|b|\frac{\pi}{2} = 2\pi$$
$$|b| = 4$$
$$b = \pm 4$$

Sample answer: One sinusoidal function in which a = 1.5 and b = 4 is y = 1.5 cos 4x. Evaluate the function for $x = \frac{\pi}{2}$.

$$y = 1.5\cos 4x$$
$$y = 1.5\cos\left(4 \cdot \frac{\pi}{2}\right)$$
$$y = 1.5\cos 2\pi$$
$$x = 1.5(1)$$
$$x = 1.5 \text{ or } \frac{3}{2}$$

The function passes through $\left(\frac{\pi}{2}, \frac{3}{2}\right)$. Therefore, a sinusoidal function with period $\frac{\pi}{2}$ and amplitude 1.5 that passes through the point $\left(\frac{\pi}{2}, \frac{3}{2}\right)$ is y = 1.5 cos 4x.

38. period: 3π; amplitude: $\frac{1}{2}$; point: $\left(\pi, \frac{\sqrt{3}}{4}\right)$

SOLUTION:

Use the period to find b.

$$3\pi = \frac{2\pi}{|b|}$$
$$|b|3\pi = 2\pi$$
$$|b| = \frac{2}{3}$$
$$b = \pm\frac{2}{3}$$

Sample answer: One sinusoidal function in which a = 0.5 and $b = \frac{2}{3}$ is $y = \frac{1}{2} \sin \frac{2}{3}x$. Evaluate the function for $x = \pi$.

$$y = \frac{1}{2}\sin\frac{2}{3}x$$
$$y = \frac{1}{2}\sin\left(\frac{2}{3} \cdot \pi\right)$$
$$y = \frac{1}{2}\sin\frac{2\pi}{3}$$
$$x = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$
$$x = \frac{\sqrt{3}}{4}$$

The function passes through $\left(\pi, \frac{\sqrt{3}}{4}\right)$. Therefore, a sinusoidal function with period 3π and amplitude 0.5 that passes through the point $\left(\pi, \frac{\sqrt{3}}{4}\right)$ is $y = \frac{1}{2} \sin \frac{2}{3}x$.

39. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the change in the graph of a sinusoidal function of the form y = sin x or y = cos x when multiplied by a polynomial function.

a. GRAPHICAL Use a graphing calculator to sketch the graphs of $y = 2x$, $y = -2x$, and $y = 2x \cos x$ on the same coordinate plane, on the interval [–20, 20].

b. VERBAL Describe the behavior of the graph of

$y = 2x \cos x$ in relation to the graphs of $y = 2x$ and $y = -2x$.

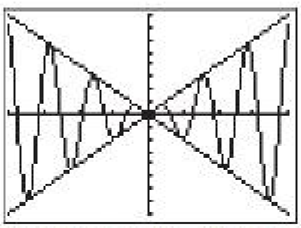
c. GRAPHICAL Use a graphing calculator to sketch the graphs of $y = x^2$, $y = -x^2$, and $y = x^2 \sin x$ on the same coordinate plane, on the interval [–20, 20].

d. VERBAL Describe the behavior of the graph of $y = x^2 \sin x$ in relation to the graphs of $y = x^2$ and $y = -x^2$.

e. ANALYTICAL Make a conjecture as to the behavior of the graph of a sinusoidal function of the form $y = \sin x$ or $y = \cos x$ when multiplied by polynomial function of the form $y = f(x)$.

SOLUTION:

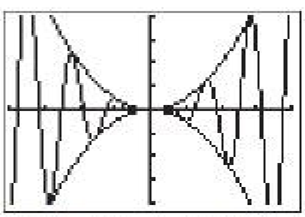
a.



[–20, 20] scl: 5 by [–40, 40] scl: 5

b. The graph of $y = 2x \cos x$ oscillates between the graphs of $y = 2x$ and $y = -2x$.

c.



[–20, 20] scl: 5 by [–200, 200] scl: 50

d. The graph of $y = x^2 \sin x$ oscillates between the graphs of $y = x^2$ and $y = -x^2$.

e. The graph of $y = f(x) \sin x$ or $y = f(x) \cos x$ will oscillate between the graphs of $y = f(x)$ and $y = -f(x)$.

40. **CHALLENGE** Without graphing, find the exact coordinates of the first maximum point to the right of the y-axis for $y = 4\sin\left(\frac{2}{3}x - \frac{\pi}{9}\right)$.

SOLUTION:

First, find the phase shift for $y = 4\sin\left(\frac{2}{3}x - \frac{\pi}{9}\right)$.

$$\begin{aligned}\text{phase shift} &= -\frac{c}{b}\\ &= -\frac{-\frac{\pi}{9}}{\frac{2}{3}}\\ &= \frac{3\pi}{18} \text{ or } \frac{\pi}{6}\end{aligned}$$

Next, find the period of the function.

$$\begin{aligned}\text{period} &= \frac{2\pi}{|b|}\\ &= \frac{2\pi}{\left|\frac{2}{3}\right|}\\ &= \frac{2\pi}{\frac{2}{3}}\\ &= \frac{6\pi}{2} \text{ or } 3\pi\end{aligned}$$

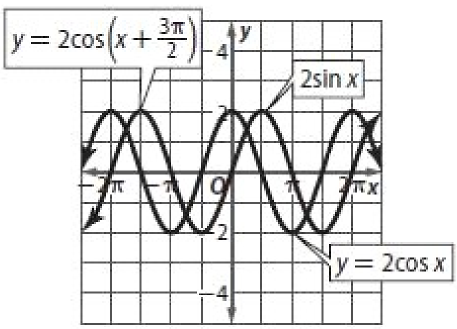
The first maximum for one period can be found by finding $\frac{\text{period}}{4}$. So, incorporating the phase shift, the first maximum to the right of the y-axis is $\frac{3\pi}{4} + \frac{\pi}{6} = \frac{11\pi}{12}$. Because the amplitude is |4| = 4, the maximum point is $\left(\frac{11\pi}{12}, 4\right)$.

REASONING Determine whether each statement is *true* or *false*. Explain your reasoning.

41. Every sine function of the form $y = a \sin(bx + c) + d$ can be written as a cosine function of the form $y = a \cos(bx + c) + d$.

SOLUTION:

It is true that every sine function of the form $y = a \sin(bx + c) + d$ can be written as a cosine function of the form $y = a \cos(bx + c) + d$.
The graph of cosine is a horizontal translation of the sine graph. Therefore, a cosine function can be written from any sine function using the same amplitude and period by applying the necessary phase shift.
For example, consider $y = 2\sin x$ and $y = 2\cos\left(x + \frac{3\pi}{2}\right)$. Phase shift $y = 2\cos x$ by $\frac{3\pi}{2}$ to obtain $y = 2\sin x$.



42. The period of $f(x) = \cos 8x$ is equal to four times the period of $g(x) = \cos 2x$.

SOLUTION:

Find the period of $f(x) = \cos 8x$.

$$\text{period} = \frac{2\pi}{|b|} = \frac{2\pi}{|8|} = \frac{\pi}{4}$$

Find the period of $g(x) = \cos 2x$.

$$\text{period} = \frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \pi$$

Four times the period of $g(x) = \cos 2x$ is $4 \cdot \pi$ or 4π. So, the period of $f(x)$ is $\frac{1}{4}$ the period of $g(x)$. Therefore, the statement is false.

4-4 Graphing Sine and Cosine Functions

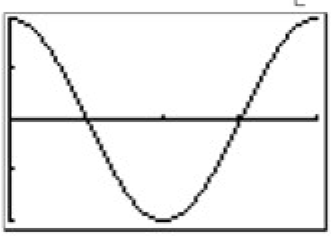
43. **CHALLENGE** How many zeros does $y = \cos 1500x$ have on the interval $0 \le x \le 2\pi$?

SOLUTION:

Find the period of $y = \cos 1500x$.

$$\text{period} = \frac{2\pi}{|b|}$$
$$= \frac{2\pi}{|1500|} \text{ or } \frac{\pi}{750}$$

Graph $y = \cos 1500x$ on $\left[0, \frac{\pi}{750}\right]$.



$\left[0, \frac{\pi}{750}\right]$ scl: $\frac{\pi}{3000}$ by $[-1, 1]$ scl: 0.5

The graph of $y = \cos 1500x$ has two x-intercepts for one cycle, so the function has 2 zeros per cycle. Find the number of cycles for $y = \cos 1500x$ on $[0, 2\pi]$.

$$2\pi = \left(\frac{\pi}{750}\right)x$$
$$\left(\frac{750}{\pi}\right)2\pi = x$$
$$1500 = x$$

So, the graph of $y = \cos 1500x$ will complete 1500 cycles on $[0, 2\pi]$. Because there are two zeros per cycle, $y = \cos 1500x$ will have 1500(2) or 3000 zeros on $[0, 2\pi]$.

44. **PROOF** Prove the phase shift formula.

SOLUTION:

Consider $y = a \sin(bx + c)$, where a, b, and $c \neq 0$. To find a zero of the function, find the value of x for which $a \sin(bx + c) = 0$. Since $\sin 0 = 0$, solving $bx + c = 0$ will yield a zero of the function.

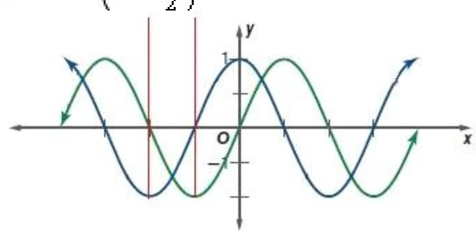
$$bx + c = 0$$
$$bx = -c$$
$$x = -\frac{c}{b}$$

Therefore, $y = 0$ when $x = -\frac{c}{b}$. The value of $-\frac{c}{b}$ is the phase shift.

When $c > 0$: The graph of $y = a \sin(bx + c)$ is the graph of $y = a \sin x$, shifted $\left|\frac{c}{b}\right|$ units to the left.

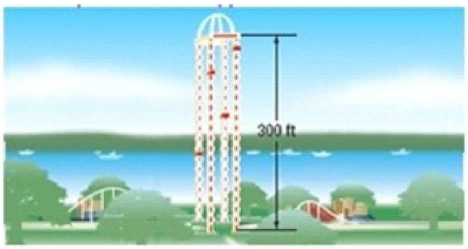
When $c < 0$: The graph of $y = a \sin(bx + c)$ is the graph of $y = a \sin x$, shifted $\left|\frac{c}{b}\right|$ units to the right.

Consider the graph of $y = \cos x$ below in blue. In this equation $a = 1$, $b = 1$ and $c = 0$. One zero of $y = \cos x$ is $\frac{\pi}{2}$. The green graph is shifted $\frac{\pi}{2}$ to the left. Thus c is $-\frac{\pi}{2}$. Then the shifted equation will be $y = \cos\left(x - \frac{\pi}{2}\right)$.



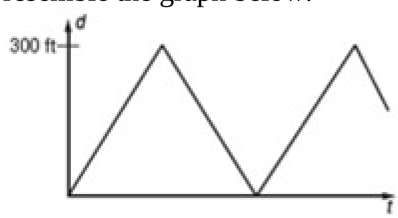
4-4 Graphing Sine and Cosine Functions

45. ***Writing in Math*** The Power Tower ride in Sandusky, Ohio, is shown below. Along the side of each tower is a string of lights that send a continuous pulse of light up and down each tower at a constant rate. Explain why the distance *d* of this light from the ground over time *t* cannot be represented by a sinusoidal function.



SOLUTION:

Sample answer: Although the pulse of light can be represented as a function with a period, it is not a sinusoidal function because the distance the pulse of light is from the ground changes at a constant rate. As a result, the graph of this function would resemble the graph below.



The given point lies on the terminal side of an angle θ in standard position. Find the values of the six trigonometric functions of θ.

46. (−4, 4)

SOLUTION:

Use the values of x and y to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-4)^2 + 4^2} \\ &= \sqrt{32} \text{ or } 4\sqrt{2} \end{aligned}$$

Use $x = -4$, $y = 4$, and $r = 4\sqrt{2}$ to write the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \cos\theta = \frac{x}{r} = -\frac{4}{4\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{4\sqrt{2}}{4} = \sqrt{2} \quad \sec\theta = \frac{r}{x} = -\frac{4\sqrt{2}}{4} = -\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -\frac{4}{4} = -1 \quad \cot\theta = \frac{x}{y} = -\frac{4}{4} = -1$$

47. (8, −2)

SOLUTION:

Use the values of x and y to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{8^2 + (-2)^2} \\ &= \sqrt{68} \text{ or } 2\sqrt{17} \end{aligned}$$

Use $x = 8$, $y = -2$, and $r = 2\sqrt{17}$ to write the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = -\frac{2}{2\sqrt{17}} = -\frac{\sqrt{17}}{17} \quad \cos\theta = \frac{x}{r} = \frac{8}{2\sqrt{17}} = \frac{4\sqrt{17}}{17}$$

$$\csc\theta = \frac{r}{y} = -\frac{2\sqrt{17}}{2} = -\sqrt{17} \quad \sec\theta = \frac{r}{x} = \frac{2\sqrt{17}}{8} = \frac{\sqrt{17}}{4}$$

$$\tan\theta = \frac{y}{x} = -\frac{2}{8} = -\frac{1}{4} \quad \cot\theta = \frac{x}{y} = -\frac{8}{2} = -4$$

48. (−5, −9)

SOLUTION:

Use the values of x and y to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + (-9)^2} \\ &= \sqrt{106} \end{aligned}$$

Use $x = -5$, $y = -9$, and $r = \sqrt{106}$ to write the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = -\frac{9}{\sqrt{106}} = -\frac{9\sqrt{106}}{106} \quad \cos\theta = \frac{x}{r} = -\frac{5}{\sqrt{106}} = -\frac{5\sqrt{106}}{106}$$

$$\csc\theta = \frac{r}{y} = -\frac{\sqrt{106}}{9} \quad \sec\theta = \frac{r}{x} = -\frac{\sqrt{106}}{5} = -\frac{\sqrt{106}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{9}{5} \quad \cot\theta = \frac{x}{y} = \frac{5}{9}$$

49. (4, 5)

SOLUTION:

Use the values of x and y to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4^2 + 5^2} \\ &= \sqrt{41} \end{aligned}$$

Use $x = 4$, $y = 5$, and $r = \sqrt{41}$ to write the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41} \quad \cos\theta = \frac{x}{r} = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{41}}{5} \quad \sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4}$$

$$\tan\theta = \frac{y}{x} = \frac{5}{4} \quad \cot\theta = \frac{x}{y} = \frac{4}{5}$$

Write each degree measure in radians as a multiple of π and each radian measure in degrees.

50. 25°

SOLUTION:

To convert a degree measure to radians, multiply by $\frac{\pi \text{ radians}}{180°}$.

$$\begin{aligned} 25° &= 25°\left(\frac{\pi \text{ radians}}{180°}\right) \\ &= \frac{5\pi}{36} \text{ radians} \\ &= \frac{5\pi}{36} \end{aligned}$$

51. –420°

SOLUTION:

To convert a degree measure to radians, multiply by $\frac{\pi \text{ radians}}{180°}$.

$$\begin{aligned} -420° &= -420°\left(\frac{\pi \text{ radians}}{180°}\right) \\ &= -\frac{7\pi}{3} \text{ radians} \\ &= -\frac{7\pi}{3} \end{aligned}$$

52. $-\frac{\pi}{4}$

SOLUTION:

To convert a radian measure to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

$$\begin{aligned} -\frac{\pi}{4} &= -\frac{\pi}{4} \text{ radians} \\ &= -\frac{\pi}{4} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right) \\ &= -\frac{180°}{4} \\ &= -45° \end{aligned}$$

53. $\frac{8\pi}{3}$

SOLUTION:

To convert a radian measure to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

$$\begin{aligned} \frac{8\pi}{3} &= \frac{8\pi}{3} \text{ radians} \\ &= \frac{8\pi}{3} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right) \text{ or } 480° \\ &= \frac{1440°}{3}\ 480° \\ &= 480° \end{aligned}$$

54. **SCIENCE** Radiocarbon dating is a method of estimating the age of an organic material by calculating the amount of carbon-14 present in the material. The age of a material can be calculated using $A = t\,\frac{\ln R}{-0.693}$, where A is the age of the object in years, t is the half-life of carbon-14 or 5700 years, and R is the ratio of the amount of carbon-14 in the sample to the amount of carbon-14 in living tissue.

a. A sample of organic material contains 0.000076 gram of carbon-14. A living sample of the same material contains 0.00038 gram. About how old is the sample?

b. A specific sample is at least 20,000 years old. What is the maximum percent of carbon-14 remaining in the sample?

SOLUTION:

a. Use the formula $A = t \cdot \frac{\ln R}{-0.693}$ to find the age of the sample. The half-life t is 5700. R is the ratio of the amount in the sample to the amount in living tissue. Divide to find R.

$$R = \frac{0.000076}{0.00038} = 0.2$$

Substitute the values into the formula to find A.

$$A = t \cdot \frac{\ln R}{-0.693}$$
$$A = 5700 \cdot \frac{\ln 0.2}{-0.693}$$
$$A \approx 13{,}237.8$$

The sample is about 13,238 years old.

b. Set up an inequality using A = 20,000.

$$A \ge t \cdot \frac{\ln R}{-0.693}$$
$$20{,}000 \ge 5700 \cdot \frac{\ln R}{-0.693}$$
$$\frac{200}{57} \ge \frac{\ln R}{-0.693}$$
$$-0.693\left(\frac{200}{57}\right) \ge \ln R$$
$$e^{-0.693\left(\frac{200}{57}\right)} \ge R$$
$$0.0879 \ge R$$

The maximum percent of carbon in the sample is approximately 8.8%.

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

55. $f(x) = x^3 + 2x^2 - 8x$

SOLUTION:

The degree of $f(x)$ is 3, so it will have at most three real zeros and two turning points.

$$0 = x^3 + 2x^2 - 8x$$
$$0 = x(x^2 + 2x - 8)$$
$$0 = x(x + 4)(x - 2)$$

So, the zeros are −4, 0, and 2.

56. $f(x) = x^4 - 10x^2 + 9$

SOLUTION:

The degree of $f(x)$ is 4, so it will have at most four real zeros and three turning points.

$$0 = x^4 - 10x^2 + 9$$
$$0 = (x^2 - 9)(x^2 - 1)$$
$$0 = (x + 3)(x - 3)(x + 1)(x - 1)$$

So, the zeros are −3, −1, 1, and 3.

57. $f(x) = x^5 + 2x^4 - 4x^3 - 8x^2$

SOLUTION:

The degree of $f(x)$ is 5, so it will have at most five real zeros and four turning points.

$$0 = x^5 + 2x^4 - 4x^3 - 8x^2$$
$$0 = x^2(x^3 + 2x^2 - 4x - 8)$$
$$0 = x^2[x^2(x + 2) + (-4)(x + 2)]$$
$$0 = x^2(x + 2)(x^2 - 4)$$
$$0 = x^2(x + 2)(x^2 - 4)$$
$$0 = x^2(x + 2)(x - 2)(x + 2)$$

So, the zeros are −2, 0, and 2.

58. $f(x) = x^4 - 1$

SOLUTION:

The degree of $f(x)$ is 4, so it will have at most four real zeros and five turning points.

$$0 = x^4 - 1$$
$$0 = (x^2 - 1)(x^2 + 1)$$
$$0 = (x - 1)(x + 1)(x^2 + 1)$$

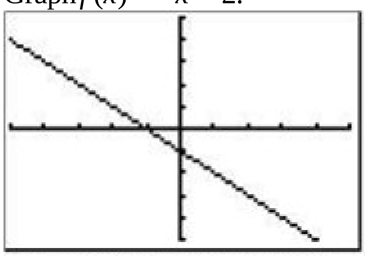
So, the zeros are −1, and 1.

Determine whether f has an inverse function. If it does, find the inverse function and state any restrictions on its domain.

59. $f(x) = -x - 2$

SOLUTION:

Graph $f(x) = -x - 2$.



[–10, 10] scl: 2 by [–10, 10] scl: 2

The graph of f passes the horizontal line test. Therefore, f is a one-to-one function and has an inverse function. From the graph, you can see that f has domain $(-\infty, \infty)$ and range $(-\infty, \infty)$. Now find f^{-1}.

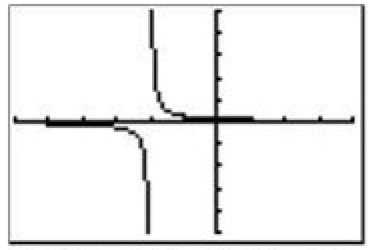
$$\begin{aligned} f(x) &= -x - 2 \\ y &= -x - 2 \\ x &= -y - 2 \\ x + y &= -2 \\ y &= -x - 2 \\ f^{-1}(x) &= -x - 2 \end{aligned}$$

Because f^{-1} and f are the same function, the domain and range of f are equal to the domain and range of f^{-1}, respectively. Therefore, it is not necessary to restrict the domain of f^{-1}.

60. $f(x) = \dfrac{1}{x+4}$

SOLUTION:

Graph $f(x) = \dfrac{1}{x+4}$.

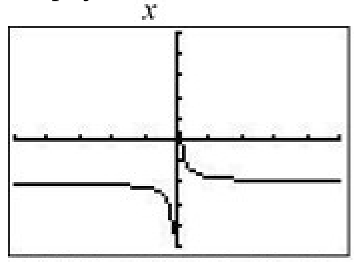


[–12, 8] scl: 2 by [–10, 10] scl: 2

The graph of f passes the horizontal line test. Therefore, f is a one-to-one function and has an inverse function. From the graph, you can see that f has domain $(-\infty, -4) \cup (-4, \infty)$ and range $(-\infty, 0) \cup (0, \infty)$. Now find f^{-1}.

$$\begin{aligned} f(x) &= \frac{1}{x+4} \\ y &= \frac{1}{x+4} \\ x &= \frac{1}{y+4} \\ x(y+4) &= 1 \\ y + 4 &= \frac{1}{x} \\ y &= \frac{1}{x} - 4 \\ f^{-1}(x) &= \frac{1}{x} - 4, \text{ where } x \neq 0 \end{aligned}$$

Graph $y = \dfrac{1}{x} - 4$.



[–10, 10] scl: 2 by [–10, 10] scl: 2

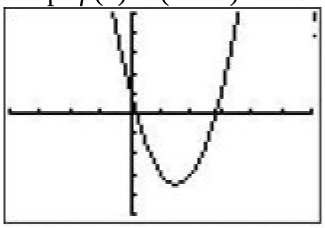
From the graph of $y = \dfrac{1}{x} - 4$ shown, you can see that the inverse relation has domain $(-\infty, 0) \cup (0, \infty)$ and range $(-\infty, -4) \cup (-4, \infty)$. The domain and range of f are equal to the domain and range of f^{-1}, respectively. Therefore, it is not necessary to restrict the domain of f^{-1}.

4-4 Graphing Sine and Cosine Functions

61. $f(x) = (x - 3)^2 - 7$

SOLUTION:

Graph $f(x) = (x - 3)^2 - 7$.



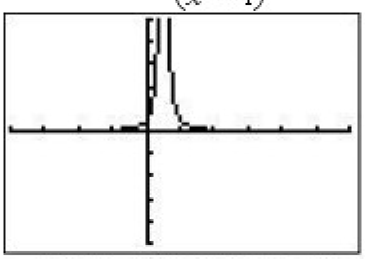
[–8, 12] scl: 2 by [–10, 10] scl: 2

The graph of f does not pass the horizontal line test. Therefore, f is not a one-to-one function and does not have an inverse function.

62. $f(x) = \dfrac{1}{(x-1)^2}$

SOLUTION:

Graph $f(x) = \dfrac{1}{(x-1)^2}$.



[–8, 12] scl: 2 by [–10, 10] scl: 2

The graph of f does not pass the horizontal line test. Therefore, f is not a one-to-one function and does not have an inverse function.

63. **SAT/ACT** If $x + y = 90°$ and x and y are both nonnegative angles, what is equal to $\dfrac{\cos x}{\sin y}$?

A 0
B $\dfrac{1}{2}$
C 1
D 1.5
E Cannot be determined from the information given.

SOLUTION:

If $x + y = 90°$, then $x = 90° - y$, and $\dfrac{\cos x}{\sin y}$ can be rewritten as $\dfrac{\cos(90° - y)}{\sin y}$. Let $y = 60°$.

$$\begin{aligned}\frac{\cos(90°-y)}{\sin y} &= \frac{\cos(90°-60°)}{\sin 60°}\\ &= \frac{\cos 30°}{\sin 60°}\\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}\\ &= 1\end{aligned}$$

Try a second angle. Let $y = 45°$.

$$\begin{aligned}\frac{\cos(90°-y)}{\sin y} &= \frac{\cos(90°-45°)}{\sin 45°}\\ &= \frac{\cos 45°}{\sin 45°}\\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\\ &= 1\end{aligned}$$

Therefore, $\dfrac{\cos x}{\sin y} = 1$, and the correct answer is C.

64. **REVIEW** If $\tan x = \frac{10}{24}$ in the figure below, what are $\sin x$ and $\cos x$?

F $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$

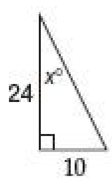
G $\sin x = \frac{10}{26}$ and $\cos x = \frac{24}{26}$

H $\sin x = \frac{26}{10}$ and $\cos x = \frac{26}{24}$

J $\sin x = \frac{10}{26}$ and $\cos x = \frac{26}{24}$

SOLUTION:

If $\tan x = \frac{10}{24}$, then the side opposite x is 10 and the side adjacent to x is 24.



Use the Pythagorean Theorem to find the length of the hypotenuse.

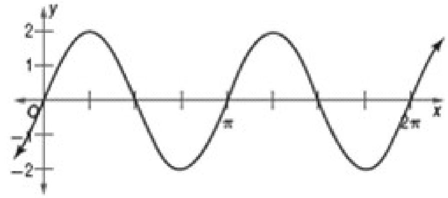
$$\begin{aligned}\text{hyp} &= \sqrt{24^2 + 10^2}\\ &= \sqrt{676}\\ &= 26\end{aligned}$$

Find $\sin x$ and $\cos x$.

$$\begin{aligned}\sin x &= \frac{\text{opp}}{\text{hyp}}\\ &= \frac{10}{26}\end{aligned} \qquad \begin{aligned}\cos x &= \frac{\text{adj}}{\text{hyp}}\\ &= \frac{24}{26}\end{aligned}$$

Therefore, the correct answer is G.

65. Identify the equation represented by the graph.



A $y = \frac{1}{2} \sin 4x$

B $y = \frac{1}{4} \sin 2x$

C $y = 2 \sin 2x$

D $y = 4 \sin \frac{1}{2} x$

SOLUTION:

From the graph, it appears that the amplitude is 2 and the period is π. Use the period to find b.

$$\begin{aligned}\text{period} &= \frac{2\pi}{|b|}\\ \pi &= \frac{2\pi}{|b|}\\ |b|\pi &= 2\pi\\ |b| &= 2\\ b &= \pm 2\end{aligned}$$

So, the sine function represented by the graph could be $y = 2 \sin 2x$. Therefore, the correct answer is C.

66. **REVIEW** If $\cos\theta = \frac{8}{17}$ and the terminal side of the angle is in Quadrant IV, what is the exact value of $\sin\theta$?

F $-\frac{15}{8}$

G $-\frac{17}{15}$

H $-\frac{15}{17}$

J $-\frac{8}{15}$

SOLUTION:

The terminal side of the angle is in Quadrant IV, so x is positive and y is negative. Because $\cos\theta = \frac{x}{r} = \frac{8}{17}$, $x = 8$ and $r = 17$. Find y.

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ &= \sqrt{17^2 - 8^2} \\ &= \sqrt{225} \text{ or } 15 \end{aligned}$$

So, $y = -15$.

Therefore, $\sin\theta = \frac{y}{r} = -\frac{15}{17}$, and the correct answer is H.