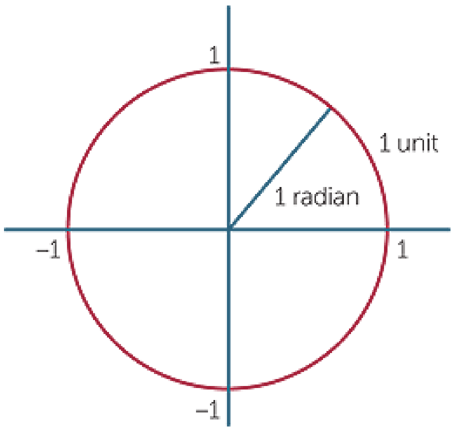
ALGEBRA OF TRIGONOMETRIC FUNCTIONS

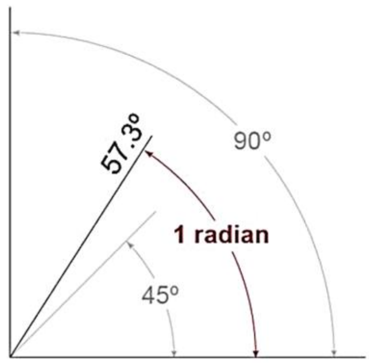
MEASURING ANGLES

Angles are usually measured in radians (c). The radian is defined as the angle that results when the length of the arc of a circle is equal to the radius of that circle.

As the circumference of a circle is $2\pi r$, there are 2π radians in a full circle.

Therefore, 1 radian $(1^c) = \left(\frac{180}{\pi}\right)^o$ and 1 degree $(1^o) = \frac{\pi}{180}$ radians.





ANGLE CONVERSIONS

Degrees $\times \frac{\pi}{180}$ → Radians

Radians $\times \frac{180}{\pi}$ → Degrees

- To convert degrees to radians, multiply by $\frac{\pi^c}{180^o}$.

  For example: $160^o = 160^0 \times \frac{\pi}{180^0} = \frac{8\pi^c}{9}$

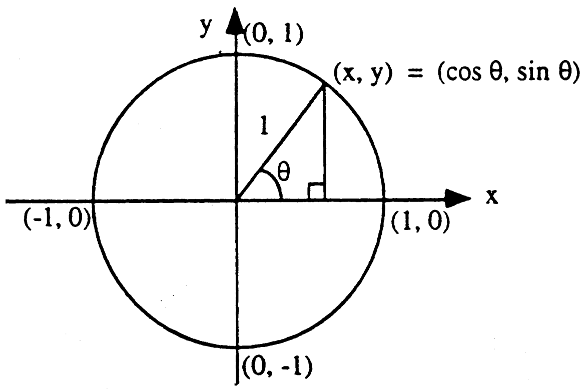
- To convert radians to degrees, multiply by $\frac{180^o}{\pi^c}$.

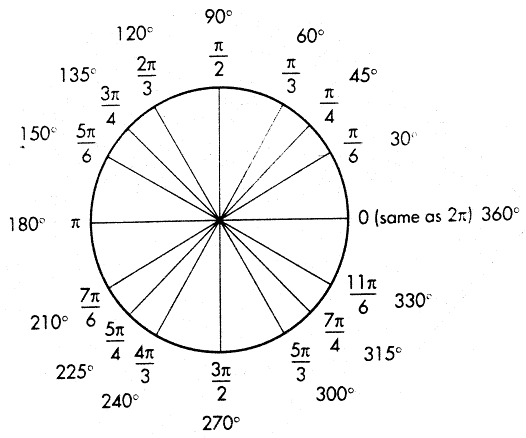
  For example: $\frac{4\pi}{3} = \frac{4\pi}{3} \times \frac{180}{\pi} = 240^0$

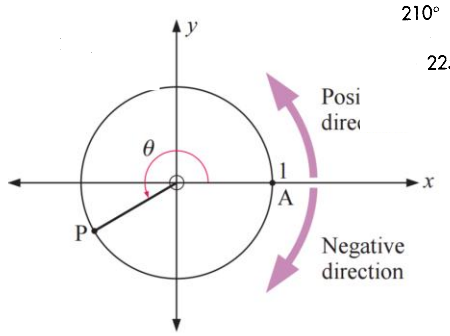
THE UNIT CIRCLE

The principles and definitions in trigonometry are based on the **UNIT** circle which is a circle with centre $(0,0)$ and a radius of 1 unit. The equation of this circle is $x^2 + y^2 = 1$.

Domain: $-1 \le x \le 1$
Range: $-1 \le y \le 1$







If θ is an angle measured anti clockwise from the positive direction of the X axis:

$\sin\theta$ represents the y coordinate of a point $P(\theta)$ on the unit circle.

For example: $\sin\left(\frac{3\pi}{2}\right) = y\ coordinate at\ 270^0 = -1$

$\cos\theta$ represents the x coordinate of a point $P(\theta)$ on the unit circle.

For example: $\cos\left(\frac{\pi}{2}\right) = x\ coordinate at\ 90^0 = 0$

$\tan\theta$ represents the gradient of the radius line that passes through a point $P(\theta)$ that lies on the circle $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$.

For example: $\tan\left(\frac{\pi}{2}\right) = \frac{y\ coordinate at\ 90^0}{x\ coordinate at\ 90^0} = \frac{1}{0} = undefined$

ANGLES AT THE AXES

$\sin(0)=0$	$\cos(0)=1$	$\tan(0)=0$
$\sin\left(\frac{\pi}{2}\right)=1$	$\cos\left(\frac{\pi}{2}\right)=0$	$\tan\left(\frac{\pi}{2}\right)=undefined$
$\sin(\pi)=0$	$\cos(\pi)=-1$	$\tan(\pi)=0$
$\sin\left(\frac{3\pi}{2}\right)=-1$	$\cos\left(\frac{3\pi}{2}\right)=0$	$\tan\left(\frac{3\pi}{2}\right)=undefined$
$\sin(2\pi)=0$	$\cos(2\pi)=1$	$\tan(2\pi)=0$

NOTE:

Positive angles – Move in an anticlockwise direction from the positive X axis.

Negative angles – Move in a clockwise direction from the positive X axis.

For example: $\cos\left(-\frac{\pi}{2}\right)= x\ coordinateat\ 270^0 = 0$

RECIPROCAL FUNCTIONS (NOT EXAMINABLE)

$\sec\theta=\frac{1}{\cos\theta}$ $\quad co\sec\theta=\frac{1}{\sin\theta}$ $\quad \cot\theta=\frac{1}{\tan\theta}$

WATCHOUTS

- As the radius of the unit circle is one, the maximum and minimum values of $\sin\theta$ and $\cos\theta$ are ±1 i.e. All values of $\sin\theta$ and $\cos\theta$ must lie between -1 and 1 inclusive.

  For example: $\sin\theta=2$ does not exist
  (There is no point on the unit circle where the value of y is 2).

- Tangents can assume values between $-\infty$ and $+\infty$. $\tan\theta=$ any real number.

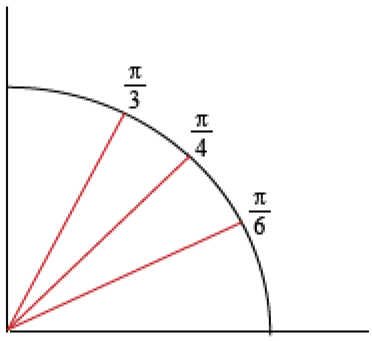
EXACT VALUES IN THE FIRST QUADRANT

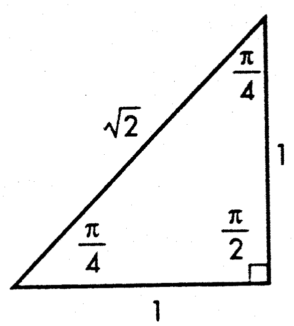
- Angles in the first quadrant are referred to as '**Reference Angles'**.

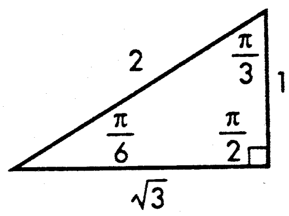
  Therefore, $0 < Reference\ Angle < \frac{\pi}{2}$

- Exact values for $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ are determined by using trigonometric ratios i.e. **SOHCAHTOA.**

$\sin\theta = \frac{opposite}{hypothenuse}$ $\cos\theta = \frac{adjacent}{hypothenuse}$ $\tan\theta = \frac{opposite}{adjacent}$







The values of important Reference Angles are given below:

Angle (θ)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos\theta$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\frac{\sqrt{3}}{1}$	Undefined

Note: In the first examination paper for Mathematical Methods, the Examiners will assume that the exact values above are known. Learn these values off by heart.

SYMMETRY PROPERTIES OF THE UNIT CIRCLE

SUPPLEMENTARY ANGLES

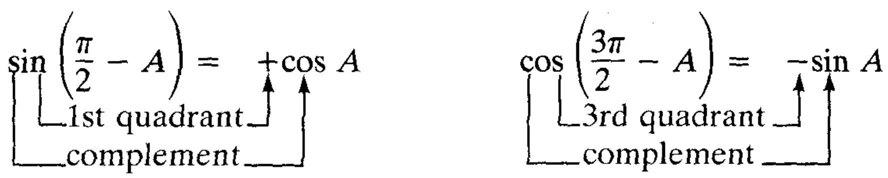
To simplify trigonometric expressions with angles $\pi \pm \theta$ or $2\pi \pm \theta$ (where θ represents an acute angle), we apply the following **supplementary rules**.

1st Quadrant	2nd Quadrant	3rd Quadrant	4th Quadrant
$\sin\theta$	$\sin(\pi-\theta)=\sin\theta$	$\sin(\pi+\theta)=-\sin\theta$	$\sin(2\pi-\theta)=\sin(-\theta)=-\sin\theta$
$\cos\theta$	$\cos(\pi-\theta)=-\cos\theta$	$\cos(\pi+\theta)=-\cos\theta$	$\cos(2\pi-\theta)=\cos(-\theta)=\cos\theta$
$\tan\theta$	$\tan(\pi-\theta)=-\tan\theta$	$\tan(\pi+\theta)=\tan\theta$	$\tan(2\pi-\theta)=\tan(-\theta)=-\tan\theta$

COMPLEMENTARY ANGLES

To simplify trigonometric expressions with angles $\frac{\pi}{2} \pm \theta$ or $\frac{3\pi}{2} \pm \theta$ (where θ represents an acute angle), we apply the following **complementary rules**.

1st Quadrant	2nd Quadrant	3rd Quadrant	4th Quadrant
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$	$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$	$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$	$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$
$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$	$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$	$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$	$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$



Note:

Complementary angles may also be used to convert both mixtures of trigonometric functions with different angles to the same trigonometric expression with different angles. These expressions can then be solved by **EQUATING** angles.

For example: $\sin(3x) = \cos\left(x + \frac{\pi}{4}\right)$

$$\therefore \cos\left(\frac{\pi}{2} - 3x\right) = \cos\left(x + \frac{\pi}{4}\right)$$

$$\cos\left(\frac{\pi}{2} - 3x\right) = \cos\left(x + \frac{\pi}{4}\right)$$

Equating angles gives $\left(\frac{\pi}{2} - 3x\right) = \left(x + \frac{\pi}{4}\right)$

$\therefore 4x = \frac{\pi}{4}$ $\quad\quad$ $\therefore x = \frac{\pi}{16} + nT$

Note: Using complementary and supplementary rules, we can write $\cos(x)$ in 6 different ways.

$$\cos(x) = \cos(-x)$$
$$\cos(x) = \cos(-x) = \cos(2\pi - x)$$

As $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ then $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

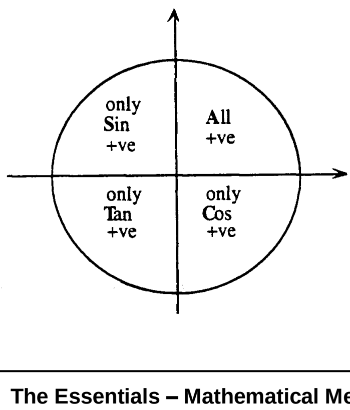
As $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$ then $\cos(x) = \sin\left(\frac{\pi}{2} + x\right)$

As $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ then $\cos(\theta) = -\sin\left(\frac{3\pi}{2} - \theta\right)$ i.e. $\cos(x) = -\sin\left(\frac{3\pi}{2} - x\right)$

As $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$ then $\cos(\theta) = -\sin\left(\frac{3\pi}{2} + \theta\right)$ i.e. $\cos(x) = -\sin\left(\frac{3\pi}{2} + x\right)$

i.e. We can use complementary and supplementary angles to write any trigonometric expressions in different ways.

SUMMARY OF TRIGONOMETRIC SIGNS (CAST)



FINDING EXACT VALUES OF TRIGONOMETRIC EXPRESSIONS

METHOD:

Step 1: Identify the quadrant in which the angle lies.

Step 2: Write the given expression in terms of a 1st quadrant angle.

Step 3: Write the appropriate quadrant rule and solve.

To write an expression whose angle is based on $\begin{pmatrix} \pi \pm \theta \\ 2\pi \pm \theta \end{pmatrix}$ in terms of a first quadrant angle, apply the following rule:

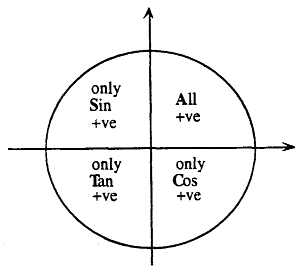
$$\begin{matrix} \sin \\ \cos \\ \tan \end{matrix} \begin{pmatrix} \pi \pm \theta \\ 2\pi \pm \theta \end{pmatrix} = \pm \begin{matrix} \sin \\ \cos \\ \tan \end{matrix} \quad (\theta)$$

Use CAST to determine the sign of the answer.

To write an expression whose angle is based on $\begin{pmatrix} \frac{\pi}{2} \pm \theta \\ \frac{3\pi}{2} \pm \theta \end{pmatrix}$ in terms of a first quadrant angle, apply the following rule:

$$\begin{matrix} \sin \\ \cos \end{matrix} \begin{pmatrix} \frac{\pi}{2} \pm \theta \\ \frac{3\pi}{2} \pm \theta \end{pmatrix} = \pm \begin{matrix} \cos \\ \sin \end{matrix} \quad (\theta)$$

Use CAST to determine the sign of the answer.



For example:

$$\cos\left(\frac{2\pi}{3}\right) = \cos(\pi - \theta) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\theta = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\tan\left(\frac{11\pi}{6}\right) = \tan(2\pi - \theta) = \tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\theta = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

TRIGONOMETRIC IDENTITIES

By applying the Pythagorean Identity on the Unit Circle, the following relationship is obtained:

$$x^2 + y^2 = 1$$

Substituting $\sin\theta = y$ and $\cos\theta = x$, we obtain the following rule:

$$\sin^2\theta + \cos^2\theta = 1$$

This statement is true for all values of θ, and is known as an **identity**.

This identity can be manipulated by dividing every term by either $\sin^2\theta$ or $\cos^2\theta$.

Dividing by $\sin^2\theta$ gives: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Dividing by $\cos^2\theta$ gives: $\tan^2\theta + 1 = \sec^2\theta$

These identities may be used to find the value of one trigonometric expression (such as cos) given the value of a different trigonometric expression (such as sin).

Note: $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\quad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

We can also use triangles to help us solve these types of problems.

A knowledge of the following triads will assist in the construction of triangles:

3, 4, 5
5, 12, 13
8, 15, 17
7, 24, 25

Whichever technique is used, careful consideration must be given to the quadrant in which the solution lies. Make sure that you assign the correct sign (positive or negative) by considering **CAST**.

For example: If $\cos x = \dfrac{-2}{3}$ find $\sin x$ where $\dfrac{\pi}{2} \le x \le \pi$.

Since $\cos^2 x + \sin^2 x = 1$

$$\frac{4}{9} + \sin^2 x = 1$$

$$\sin x = \pm\frac{\sqrt{5}}{3}$$

Since $\sin x$ is positive in 2nd quadrant: $\sin x = \dfrac{\sqrt{5}}{3}$

DOUBLE ANGLE FORMULAE
(FORMAL APPLICATION NOT REQUIRED)

$$\sin(2x) = 2\sin x\cos x = \frac{2\tan x}{1+\tan^2 x}$$

For example: $2\sin 4x\cos 4x = \sin(2\times 4x) = \sin(8x)$

For example: $\sin 3x\cos 3x = \frac{1}{2}\sin\left(2\times 3x\right) = \frac{1}{2}\sin\left(6x\right)$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

For example: $\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \cos\left(2\times\frac{\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

For example: $\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} = \tan\left(2\times\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

INVERSE OPERATIONS

$\sin^{-1}$ undoes $\sin$ i.e. $\sin^{-1}(\sin x) = x$

$\cos^{-1}$ undoes $\cos$ i.e. $\cos^{-1}(\cos x) = x$

$\tan^{-1}$ undoes $\tan$ i.e. $\tan^{-1}(\tan x) = x$

SOLVING TRIGONOMETRIC EQUATIONS

Step 1: Write all expressions in terms of one trigonometric function.

Step 2: Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation.

Step 3: Use the sign in front of the constant on the right-hand side to determine the quadrants in which the solutions are to lie. (Use **CAST**)

Step 4: Calculate the first quadrant solution. If the exact value cannot be determined:

Press Inverse Sin, Cos or Tan of the number on the right-hand side of the equation (but ignore the sign).

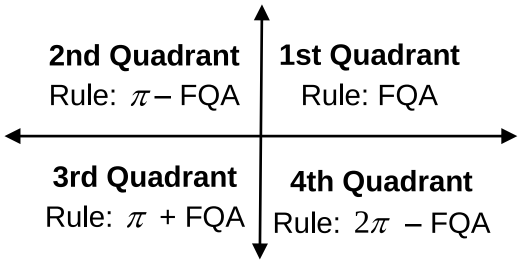
For example: $Sin^{-1}(number\ on\ RHS\ of\ equation\ but\ ignore\ the\ sign)$

(Ensure that the calculator is in Radian Mode).

Step 5: Solve for the variable (usually x or θ). Let the angle equal the rule describing angles in the quadrants in which the solutions are to lie.

Note: First Quadrant Angle = FQA

Let angle = FQA if solution lies in 1st Quadrant.
Let angle = $\pi - FQA$ if solution lies in 2nd Quadrant.
Let angle = $\pi + FQA$ if solution lies in 3rd Quadrant.
Let angle = $2\pi - FQA$ if solution lies in 4th Quadrant.



Step 6: Evaluate all possible solutions by observing the given domain. This is accomplished by adding or subtracting the **PERIOD** to each of the solutions, until the angles fall outside the given domain.

For sine and cosine functions: $Period = \dfrac{2\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

For tangent functions: $Period = \dfrac{\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

Always look closely at the brackets in the given domain and consider whether the upper and lower limits can be included in your solutions.

DO NOT discard any solution until the final step.

Step 7: Eliminate solutions that do not lie across specified domain.

Note: Students may also solve trigonometric equations by rearranging the domain.

WATCHOUTS

- Remember to add/subtract the period to each one of your solutions – making sure that you do not exceed domain. To avoid mistakes – write each term to the same denominator.
- Never ever eliminate an answer until the final step (so as to assure that all solutions are obtained).
- Look closely at brackets around the domain and assess whether the first and last solution can be included.
- Given a physical/real life situation, pay close attention to the domain. Consider the real life limitations on your solutions. For example - lengths cannot be negative.
- If number on right hand side ends up being 0 or $\pm$ 1, use the unit circle to find number of solutions per period. For all other numbers – you will get two solutions per period.
- You cannot find the inverse sine or cosine of a number greater than 1 or less than –1.

  i.e. $\sin\theta =$ number that lies between –1 and +1 inclusive.
  $\cos\theta =$ number that lies between –1 and +1 inclusive.
  $\tan\theta =$ any real number
- You cannot solve two trigonometric expressions that have different angles algebraically (use technology) unless you can find one expression in terms of the other using complementary rules.
- You cannot convert an equation containing a sin and cos to tan unless they share the same angle.
- What does $\sin 2\left(x+\frac{\pi}{3}\right)=0.5$ find???

  Answer: The points of intersection of $y=\sin 2\left(x+\frac{\pi}{3}\right)$ and $y=0.5$.
- What does $\sin 2\left(x+\frac{\pi}{3}\right)-0.5=0$ find???

  The X intercepts on the graph of $y=\sin 2\left(x+\frac{\pi}{3}\right)-0.5$.
- Given an inequation – solve the equation without the inequality and then reason from the graph.
- When solving questions to a given number of decimal places - make sure the calculator is in RADIAN mode.

SOLVING COMPLEX TRIGONOMETRIC EQUATIONS

Students are required to be able to manipulate expressions in terms of two or more different trigonometric functions, as well as solve questions involving both trigonometric functions and other expressions such as logarithmic, exponential and polynomial functions.

To solve expressions written in terms of two or more trigonometric functions, apply one of the following techniques.

If the angles are the same:

- Simplify equations by removing common factors.

  eg. $\cos^2\theta - \sin\theta\cos\theta = \cos\theta\left(\cos\theta - \sin\theta\right)$

- If the expression is presented in its factorised form (or can be factorised) and is equal to zero, apply the **null factor law** to obtain solutions.

  eg. $\cos\theta\left(\cos\theta - \sin\theta\right) = 0$

  $\therefore \cos\theta = 0 \;\; and \;\; \cos\theta - \sin\theta = 0$

- Given both a sine and cosine function – write each function on either side of the equality sign. Convert the expression to a tangent function by dividing both sides by cos or sin.

  eg. $\sin 4x + \cos 4x = 0$

  $\therefore \sin 4x = -\cos 4x$

  $\therefore \dfrac{\sin 4x}{\cos 4x} = -\dfrac{\cos 4x}{\cos 4x}$

  $\therefore \tan 4x = -1$

- Given two or more terms involving the same trigonometric function (but each with different powers) apply quadratic factors ("Let A =" method)**.**

  eg. $3\sin^6(5x) - 2\sin^3(5x) - 1 \quad \therefore Let\ A = \sin^3(5x)$

  $\therefore 3\left(\sin^3(5x)\right)^2 - 2\sin^3(5x) - 1 \quad \therefore 3A^2 - 2A - 1$

These techniques can only be successfully applied at this level of mathematics if the angles of each of the trigonometric expressions are identical.

If the angles are different:

- Use complementary or supplementary rules to write one angle in terms of the other or to write mixtures of trigonometric functions with different angles to the same trigonometric expression with different angles. These expressions can then be solved by **EQUATING** angles. eg. $\sin(3x)=\cos\left(x+\frac{\pi}{4}\right)$

$$\therefore \cos\left(\frac{\pi}{2}-3x\right)=\cos\left(x+\frac{\pi}{4}\right)$$

$$\left(\frac{\pi}{2}-3x\right)=\left(x+\frac{\pi}{4}\right)$$

$$\therefore 4x=\frac{\pi}{4}$$

$$\therefore x=\frac{\pi}{16}+nT$$

- Otherwise, use the **SOLVE** or **INTERSECT** function on your calculator to find solutions.

To solve questions involving both trigonometric functions and other expressions such as logarithmic, exponential and polynomial functions:

Use the **SOLVE** or **INTERSECT** function on your calculator to find solutions.

For example: Solve $2\sin\left(x-\frac{\pi}{2}\right)=3e^{6x-1}$.

ALGEBRA OF TRIGONOMETRIC FUNCTIONS

QUESTION 101

Show that the exact value of $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Solution

Identify the quadrant in which the angle lies: $\frac{5\pi}{4} \times \frac{180}{\pi} = 225^o$

As 225° lies between 180° and 270°, the angle lies in the third quadrant.

Write the given expression in terms of a 1st quadrant angle: $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$

Write the appropriate quadrant rule and solve:

$$\begin{matrix}\sin \\ \cos \\ \tan\end{matrix} \begin{pmatrix}\pi \pm \theta \\ 2\pi \pm \theta\end{pmatrix} = \pm \begin{matrix}\sin \\ \cos \\ \tan\end{matrix} \quad (\theta)$$

Use CAST to determine the sign of

$$\sin(\pi + \theta) = -\sin\theta \qquad \therefore \ \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

QUESTION 102

If $\cos\theta = \frac{\sqrt{3}}{4}$ and $\sin\phi = \frac{1}{\sqrt{5}}$, find the exact value of the following expressions:

(a) $\cos(\pi - \theta) = -\cos\theta = -\frac{\sqrt{3}}{4}$

(b) $\sin(\pi + \phi) = -\sin\phi = -\frac{1}{\sqrt{5}}$

(c) $\tan(2\pi - \theta) = \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} = \frac{-\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{5}}}{\frac{\sqrt{3}}{4}} = -\frac{4}{\sqrt{15}}$

(d) $\cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi = -\frac{1}{\sqrt{5}}$

QUESTION 103

Write $3\sin\left(\frac{\pi}{2}-\theta\right)+2\cos\theta$ in the form $a\cos\theta$ and hence state the value of a.

Solution

QUESTION 104

(a) If $\cos(x)=\cos\left(\frac{\pi}{3}\right)$ and $\frac{3\pi}{2}<x<2\pi$, find the value of x.

(b) If $\cos(x) = -\cos\left(\frac{\pi}{3}\right)$ and $\pi > x > \frac{\pi}{2}$, find the value of x.

Note: $\pi > x > \frac{\pi}{2}$ is equivalent to $\frac{\pi}{2} < x < \pi$.

(c) If $\tan(x) = \tan\left(\frac{\pi}{4}\right)$ and $\pi < x < \frac{3\pi}{2}$, find the value of x.

QUESTION 105

Calculate the exact value of the following expressions:

(a) $8\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right)$

(b) $1-2\sin^2\left(\frac{7\pi}{6}\right)$

QUESTION 106

Show that $\sin\theta\tan\theta = \frac{1-\cos^2\theta}{\cos\theta}$.

Solution

QUESTION 107

Write $\frac{2\cos^2\theta}{1-\sin\theta}$ in the form $a+b\sin\theta$ and hence state the value of a and b.

Solution

QUESTION 108

If $\cos\theta = -\frac{15}{17}$ and $\frac{\pi}{2} < \theta < \pi$ find $\sin\theta$.

Solution

Step 1: Use the appropriate identity to find a solution for the unknown trigonometric expression.

As $\sin^2\theta + \cos^2\theta = 1$ $\quad \therefore \sin^2\theta + \left(-\frac{15}{17}\right)^2 = 1$

$$\sin^2\theta + \frac{225}{289} = 1$$

$$\sin^2\theta = \frac{64}{289} \qquad \therefore \sin\theta = \pm\frac{8}{17}$$

Step 2: Determine the correct sign by observing the quadrant in which the solution is to lie.

Since θ lies in the second quadrant, $\sin\theta$ is positive $\therefore \sin\theta = \frac{8}{17}$.

QUESTION 109

If $\tan\theta = -\frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$ find $\sin\theta$ and $\cos\theta$.

Solution

QUESTION 110

If $\cos x = \frac{5}{12}$, find $\sin x$ given that $\frac{3\pi}{2} \le x \le 2\pi$.

Solution

QUESTION 111

If $\sin\alpha = -\frac{3}{4}$ and $\pi < \alpha < \frac{3\pi}{2}$ find $\cos\alpha$ and $\sin 2\alpha$.

Solution

SOLVING TRIGONOMETRIC EQUATIONS

QUESTION 112

Solve $2\sin 2\left(x+\frac{\pi}{3}\right)=\sqrt{3}$, $x\in[0, 2\pi]$ across the given domain.

Solution

Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation:

$$\sin 2\left(x+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Calculate the first quadrant solution:

1st Quadrant Angle $= Sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

Use the sign in front of the constant on the right hand side to determine the quadrants in which the solutions are to lie:

Solutions are to lie in the quadrants where sine is positive i.e. the 1st and 2nd quadrants:

S✓	A✓
T	C

Solve for the variable (usually x). Let the actual angle in the given equation equal the quadrant rules in which the solutions are to lie.

Let $2\left(x+\frac{\pi}{3}\right)=\frac{\pi}{3}$

Let $2\left(x+\frac{\pi}{3}\right)=\pi-\frac{\pi}{3}$

$$2\left(x+\frac{\pi}{3}\right)=\frac{\pi}{3}, \frac{2\pi}{3}$$

$$\left(x+\frac{\pi}{3}\right)=\frac{\pi}{6}, \frac{2\pi}{6}$$

$$x=-\frac{\pi}{6}, 0$$

Evaluate all possible solutions by observing the given domain. Add and subtract the PERIOD to each of the solutions, until the angles fall outside the given domain:

$$T=\frac{2\pi}{2}=\pi=\frac{6\pi}{6}$$

$$\left\{x: x=0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi\right\}$$

QUESTION 113

Which of the following gives the possible solution(s) to the equation $\sin(2x) = \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$?

I $\frac{\pi}{6}$ II $\frac{\pi}{3}$ III $\frac{\pi}{8}$ IV $\frac{\pi}{12}$

A I only
B II only
C III only
D I and II
E IV only

Solution

QUESTION 114

Find exact solutions of $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)+1=0$ for $x\in[2\pi, 6\pi]$.

Solution

QUESTION 115

Four solutions to $2\cos\left(\frac{x}{a}\right)+\sqrt{3}=0$, where $1<a<3$, are

A $x=-\frac{11\pi a}{6}$ $x=-\frac{\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{23\pi a}{6}$

B $x=-\frac{7\pi a}{6}$, $x=-\frac{5\pi a}{6}$ $x=\frac{17\pi a}{6}$ and $x=\frac{19\pi a}{6}$

C $x=-\frac{13\pi a}{6}$ $x=-\frac{11\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{23\pi a}{6}$

D $x=-\frac{19\pi a}{6}$ $x=-\frac{13\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{11\pi a}{6}$

E $x=-\frac{17\pi a}{6}$ $x=-\frac{\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{17\pi a}{6}$

Solution

QUESTION 116

A solution of the equation $\sin(3x)=k\cos(3x)$ is $\frac{3\pi}{4}$. The value of k is:

A -2

B -1

C 0

D 1

E 2

Solution

QUESTION 117

The equation $a\cos(x+b)=c,$ where a, b, c are positive constants, will not have any solutions in the interval $[0, 2\pi)$ provided that:

A $c=a$

B $b<\frac{\pi}{2}$

C $c>1$

D $a<c$

E $b>a$

Solution

QUESTION 118

Let $f(x) = a\sin(2x)$ and $g(x) = b$, where $0 < x < \frac{3\pi}{2}$ and a and b are positive integers.

Which of the following statements is **not** true?

A   If $b > a$ there are no real solutions to the equation $f(x) = g(x)$.

B   If $0 < b < a$ there are 4 solutions to the equation $f(x) = g(x)$.

C   If $0 < b < a$ there are 4 solutions to the equation $f(x) = -g(x)$.

D   If $a = b$ there is 1 solution to the equation $f(x) = -g(x)$.

E   If $a = b$ there are 2 solutions to the equation $f(x) = g(x)$.

Solution

QUESTION 119

Find exact solutions of $\tan 2\left(x+\frac{\pi}{6}\right)=\sqrt{3}$ for $x\in[-2\pi, 2\pi)$.

Solution

QUESTION 120

Find the smallest positive value of k so that the equation $\sin(4\pi x) = k\cos(4\pi x)$ will have no solution over the domain $[0, 0.1]$. State your answer correct to 4 decimal places.

Solution

QUESTION 121

Given that $\cos 2x = 0.5\sin 3x$, $x \in [0, 2\pi]$, find the solution(s) for x across the given domain. State your answer(s) correct to 3 decimal places.

Solution

QUESTION 122

Solve $2\sin\left(x - \frac{\pi}{2}\right) = -0.5e^{6x-1}$ for $[0, 2\pi]$. State your answer(s) correct to 3 decimal places.

Solution

QUESTION 123

Using algebra, show that the solution to $\cos^2 x - 2\cos x + 1 = 0$, $[0,\ 2\pi]$ is $x = 0$ and $x = 2\pi$.

Solution

QUESTION 124

Without direct substitution, show that $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$ are the exact solutions for $2\sin^2 x + 7\cos x + 2 = 0$ over the domain $[0, 2\pi]$.

Solution

QUESTION 125 – CHALLENGING QUESTION

Given that $\cos x = -\sin\left(\frac{\pi}{6}\right)$, $\pi < x < \frac{3\pi}{2}$, find x without evaluating $-\sin\left(\frac{\pi}{6}\right)$.

Solution

Solve by converting both sides to the same trigonometric function and equating angles.

To convert $\cos$ to $\sin$ write $\cos x$ as: $\sin\left(\frac{\pi}{2} - x\right)$ or

$\sin\left(\frac{\pi}{2} + x\right)$ or

$-\sin\left(\frac{3\pi}{2} - x\right)$ or

$-\sin\left(\frac{3\pi}{2} + x\right)$

To convert $\sin$ to $\cos$ write $-\sin\left(\frac{\pi}{6}\right)$ as:

$-\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ as $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ or

$\cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ as $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ or

$\cos\left(\frac{3\pi}{2} - \frac{\pi}{6}\right)$ as $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$ or

$-\cos\left(\frac{3\pi}{2} + \frac{\pi}{6}\right)$ as $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

Converting $\cos$ to $\sin$ gives: $\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(\frac{\pi}{6}\right)$

As $\sin(-x) = -\sin(x)$ then $-\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(-\frac{\pi}{6}\right)$

Equating angles: $\left(\frac{\pi}{2} - x\right) = -\frac{\pi}{6}$

$$-x = -\frac{\pi}{6} - \frac{\pi}{2}$$

$$-x = -\frac{4\pi}{6}$$

$$x = \frac{2\pi}{3}$$

Period for this expression: $\left(\frac{\pi}{2} - x\right) = -\frac{\pi}{6}$

$$\left(\frac{\pi}{2} - x\right) + \frac{\pi}{6} = 0$$

$$\left(x - \frac{2\pi}{3}\right) = 0$$

$$T = \frac{2\pi}{1} = 2\pi$$

This answer is one of a number of possible solutions. To find the remaining solutions, add/subtract the period , 2π .

$\therefore x = \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$ etc, which falls outside the given domain.

Generate a new equation and solve. Continue the process until a second solution is obtained across the given domain, $\pi < x < \frac{3\pi}{2}$.

As $\cos x = \sin\left(\frac{\pi}{2} + x\right)$

As $\sin(-x) = -\sin(x)$ then $\sin\left(\frac{\pi}{2} + x\right) = \sin\left(-\frac{\pi}{6}\right)$

Equating angles: $\left(\frac{\pi}{2} + x\right) = -\frac{\pi}{6}$

$$x = -\frac{\pi}{6} - \frac{\pi}{2}$$

$$x = -\frac{4\pi}{6} = -\frac{2\pi}{3}$$

Period for this expression: $\left(\frac{\pi}{2} + x\right) = -\frac{\pi}{6}$

$$\frac{\pi}{2} + x + \frac{\pi}{6} = 0$$

$$\left(x + \frac{2\pi}{3}\right) = 0$$

$$T = \frac{2\pi}{1} = 2\pi$$

Add/subtract the period , 2π .

$\therefore x = -\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}$, which falls within the given domain.