LIMITS AND DERIVATIVES

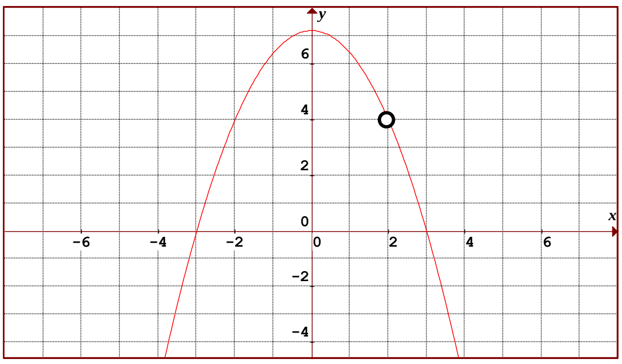
The limit of a function is the value of y that the function approaches as x gets closer to a particular value.

The limit of $f(x)$ as x approaches a is written as $\lim_{x \to a} f(x)$.

CONDITIONS FOR THE EXISTENCE OF A LIMIT

A limit may exist at $x = a$ if:

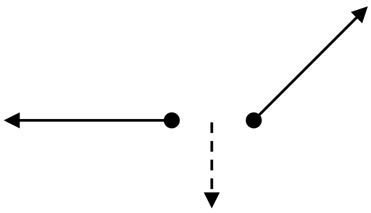
- If the function $f(x)$ is continuous at $x = a$.
- If there is point discontinuity at $x = a$.



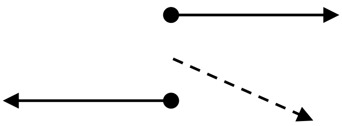
$\lim_{x \to 2} f(x)$ exists and is equal to 4.

A limit will not exist at $x = a$ if:

- There is discontinuity across an interval and $x = a$ lies within this interval.



A limit does not exist at this value of x



A limit does not exist at this value of x

EVALUATING LIMITS GRAPHICALLY

SINGLE FUNCTIONS

Step 1: Observe the value of y as x approaches a value below the point of interest.

Step 2: Observe the value of y as x approaches a value above the point of interest.

Step 3: If the left hand limit is equal to the right hand limit a limit exists at $x = a$.

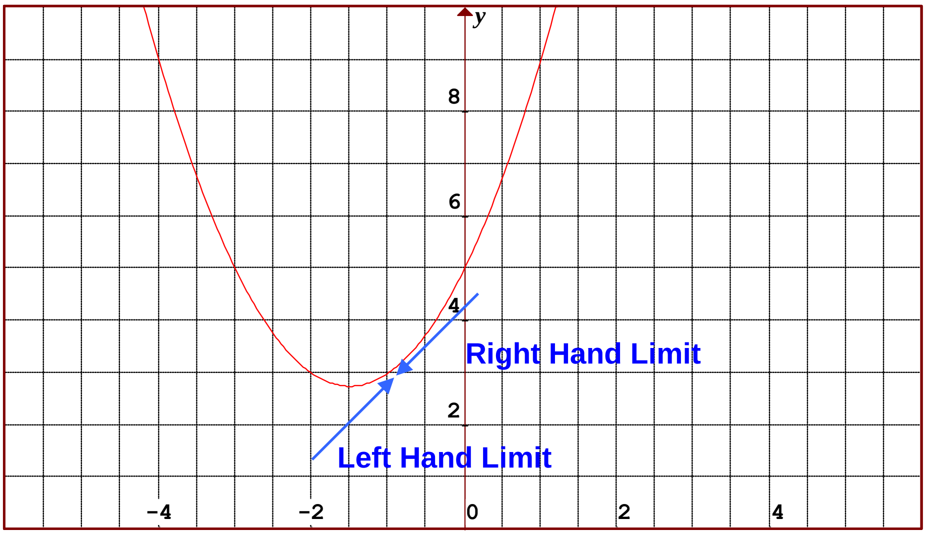
The limit is simply equal to the value of y that the curve is approaching on either side of the point of interest.

QUESTION 1 – EXAM 1

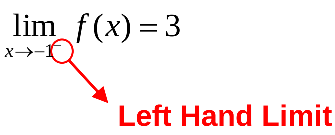
Find $\lim_{x \to -1} (x^2 + 3x + 5)$.

Solution

To find the limit, determine the value of y that the curve approaches as x gets closer to -1.



As x approaches -1 from below (or from the left hand side), $f(x)$ approaches 3.
We say that the left hand limit is equal to 3.



As x approaches -1 from above (or from the right hand side), $f(x)$ approaches 3.
We say that the right hand limit is equal to 3.

$$\lim_{x\to -1^+}(x^2+3x+5)=3$$

Right Hand Limit

When the left hand limit is equal to the right hand limit a limit exists at $x=a$. In this case, we say that the limit as x approaches -1 is 3. This is written as: $\lim_{x\to -1}(x^2+3x+5)=3$.

Note:

As the limit represents a value of y, the same answer may be obtained by simply substituting the value of a (in this case -1) into the equation describing the curve, providing the function is defined at that point.

i.e. Given $y=x^2+3x+5$

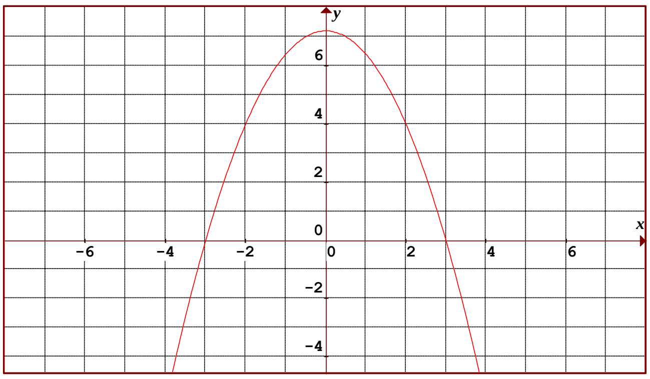
Substitute $x=-1$: $(-1)^2+3(-1)+5=3$

$\therefore \lim_{x\to -1}(x^2+3x+5)=3$

QUESTION 2 – EXAM 2

The graph of $f(x)$ is given below. $\lim_{x\to -2} f(x)$ is equal to

A 0
B -2
C 2
D 4
E Undefined



Solution

LIMITS OF HYBRID FUNCTIONS

A hybrid function is a function that has different rules describing the different sections of its domain.

Limits of hybrid functions are evaluated in the same manner as previously described.

Step 1: Find the limit of each individual function at the given value of x.

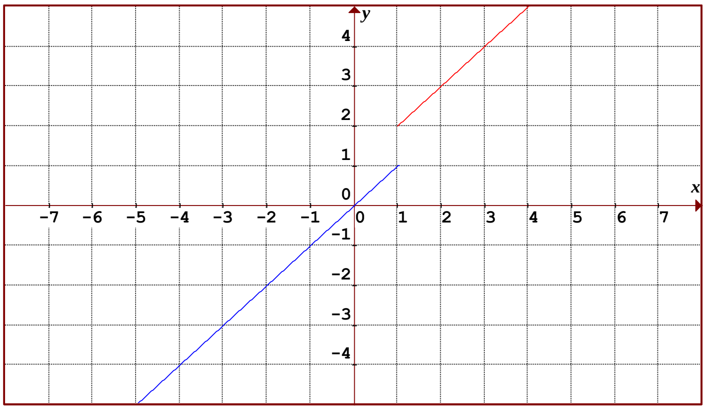
Step 2: If the limiting values are the same, the limit exists at that value of x.

If one or more of the limiting values are different, the limit does not exist at that particular value of x.

Note: If the function is discontinuous across an interval and the value of x lies in this interval, the limit cannot be evaluated

QUESTION 3 – EXAM 1

Find $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases} x+1 & \textbf{\textit{for}}\ \ x\geq 1 \\ x & \textbf{\textit{for}}\ \ x\leq 1 \end{cases}$



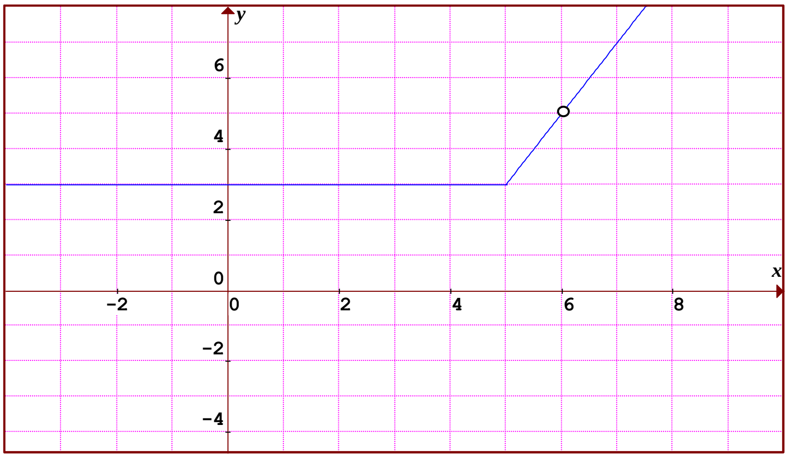
Solution

As x approaches 1 from above, $f(x)$ approaches 2.
However, as x approaches 1 from below, then $f(x)$ approaches 1.

As the two limiting values are different, we say that the $\lim_{x\to 1} f(x)$ does not exist.

QUESTION 4 – EXAM 2

The graph of $f(x)$ is given below.



Evaluate:

(a) $\lim_{x \to 5} f(x)$

(b) $\lim_{x \to 0} f(x)$

(c) $\lim_{x \to 6} f(x)$

EVALUATING LIMITS ALGEBRAICALLY

The manner in which we evaluate limits algebraically depends on whether the **denominator** of the function carries terms involving x:

Does the given expression contain a denominator bearing terms involving x?

No → Substitute the limit value into the given expression and solve.

Yes → Substitute the limit value into the denominator.

- If the denominator is equal to zero, factorise and eliminate terms by cancellation. Then substitute the limit value into the simplified expression and simplify.
- If the denominator is not equal to zero, substitute the limit value into the given expression and simplify.

IN DETAIL

Given $\lim_{x \to a} f(x)$**:**

(a) If the expression does not carry a denominator or if the denominator has **NO** terms involving x, simply substitute the value of a into the given expression and simplify.

For example: $\lim_{x \to 1}(5-2x-x^3) = 5-2(1)-(1)^3 = 2$

(b) If the denominator carries terms involving x, we need to first determine whether the value of x (or a) in question causes the function to be undefined.

Substitute the value of a into the denominator of the given expression.

- If the answer does not equal zero, substitute the value of a into the given expression.

  For example: $\lim_{x \to -2}\left(\frac{x^2+1}{x-2}\right) = \frac{(-2)^2+1}{(-2)-2} = \frac{4+1}{-4} = -\frac{5}{4}$

- If the answer is equal to zero, we need to eliminate the term(s) that is/are causing the function to become undefined. Factorise the given expression and eliminate these terms by cancellation. Before evaluating the limit, substitute the value of a into the new denominator to ensure there are no other terms present that will make the function undefined.

  For example: $\lim_{x \to 3}\left[\frac{(x^2-5x+6)}{(x-3)}\right] = \lim_{x \to 3}\left[\frac{(x-3)(x-2)}{(x-3)}\right] = \lim_{x \to 3}(x-2) = 3-2 = 1$

Note:

When there are terms involving x in the denominator of a fraction, there may be some values of x which cause the function to become undefined. These values of x must be stated with your answer.

To determine which values of x will make the function undefined, before the function is factorised and terms are eliminated by cancellation:

Step 1: Let the denominator of fraction equal zero .
Step 2: Solve for x.

For Example: $\lim_{x \to -1}\left(\frac{x^2-3x-4}{(x+1)(x+3)}\right)$

Restrictions: $x \neq -1$ or $x \neq -3$.

Note: Even though this function is undefined at $x=-1$, we may still investigate what happens to the value of y as x approaches this value (i.e. the limit).

LIMIT THEOREMS

(a) The limit of a constant is equal to the value of the constant:

If $f(x) = k$ then $\lim\limits_{x \to a} f(x) = k$.

For Example: $\lim\limits_{x \to 2} (5) = 5$

(b) The limit of the sum and/or difference of a series of terms is equal to the sum and/or difference of the limits of each individual term.

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

For Example: $\lim\limits_{x \to 2}(x^2 - 3x - 1) = \lim\limits_{x \to 2}(x^2) - \lim\limits_{x \to 2}(3x) - \lim\limits_{x \to 2}(1) = 4 - 6 - 1 = -3$

(c) The limit of the product of two functions is equal to the product of the limits of each individual function.

$$\lim_{x \to a}[f(x).g(x)] = \lim_{x \to a}(f(x)) \times \lim_{x \to a}(g(x))$$

For Example: $\lim\limits_{x \to {}^-1}(x^2 - 1)(4x) = \lim\limits_{x \to {}^-1}(x^2 - 1) \times \lim\limits_{x \to {}^-1}(4x) = 0 \times -4 = 0$

(d) The limit of the quotient of two functions is equal to the quotient of the limits of each individual function.

$$\lim\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)}$$

Note that $g(x) \neq 0$.

For Example: $\lim\limits_{x \to 3}\left(\dfrac{x - 5}{2x^2 - 9x - 5}\right) = \dfrac{\lim\limits_{x \to 3}(x - 5)}{\lim\limits_{x \to 3}(2x^2 - 9x - 5)} = \dfrac{-2}{18 - 27 - 5} = \dfrac{-2}{-14} = \dfrac{1}{7}$

OR

$$= \lim_{x \to 3}\left(\frac{(x-5)}{(x-5)(2x+1)}\right) = \lim_{x \to 3}\left(\frac{1}{(2x+1)}\right) = \frac{1}{7}$$

QUESTION 5

Given $\lim_{x \to 8} f(x) = -9$, $\lim_{x \to 8} g(x) = 2$ and $\lim_{x \to 8} h(x) = 4$, determine the following limits.

(a) $\lim_{x \to 8} \left[2f(x) - 12h(x) \right]$

(b) $\lim_{x \to 8} \left[g(x).h(x) - f(x) \right]$

(c) $\lim_{x \to 8} \left[f(x) - g(x) + h(x) \right]$

Solution

QUESTION 6 – EXAM 1

Evaluate $\lim_{x \to 3}\left(\frac{x^2+2}{x+1}\right)$.

Solution

Check if denominator is zero when x = 3 to determine if the function is defined at this point:

$x+1=3+1=4$

As the denominator does not equal zero, substitute x = 3 and evaluate the limit:

$$\lim_{x \to 3}\left(\frac{x^2+2}{x+1}\right)=\frac{3^2+2}{3+1}=\frac{11}{4}$$

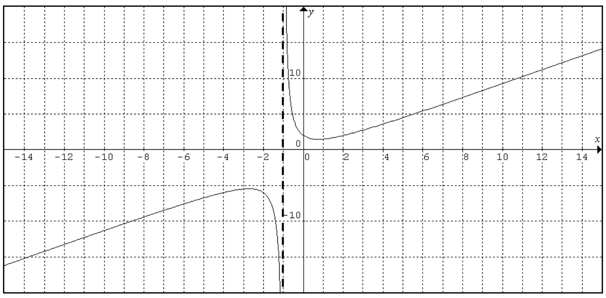
State any restrictions on the values of x:

Remember to define the restrictions using the ORIGINAL (given) equation.

$x+1 \neq 0$

$\therefore x \neq -1$

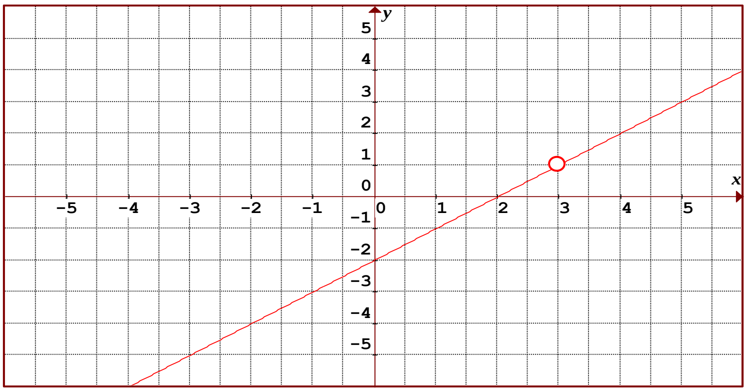
$$\therefore \lim_{x \to 3}\left(\frac{x^2+2}{x+1}\right)=\frac{11}{4}, \quad x \neq -1$$



QUESTION 7 – EXAM 1

Evaluate $\lim_{x \to 3}\left[\frac{(x^2-5x+6)}{(x-3)}\right]$.

Solution



QUESTION 8 – EXAM 1

Evaluate the following limits (if they exist).

(a) $\lim_{x \to -2}(2)$

(b) $\lim_{x \to 1}(5 - 2x - x^3)$

(c) $\lim_{x \to -3}\left(\dfrac{x^2 + 4x + 3}{x + 3}\right)$

(d) $\lim_{x \to 3}\left(\dfrac{2x - 6}{18 - 2x^2}\right)$

QUESTION 9 – EXAM 2

Evaluate the following limits (if they exist).

(a) $\lim_{x \to 2} \left(x^2 + 5 \right)$

(b) $\lim_{x \to -2} \left(\frac{x^2 + 1}{x - 2} \right)$

(c) $\lim_{x \to -2} \left(\frac{x^2 + 5x + 6}{x + 2} \right)$

(d) $\lim_{x \to 1} \left(\frac{x - 1}{x^2 + x - 2} \right)$

DIFFERENTIATION

- The derivative describes the gradient of the tangent to a curve at any value of x.

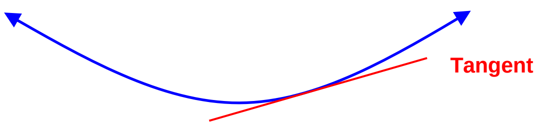
  When the equation is denoted as $f(x)$, the derivative is represented as $f'(x)$.

  When the equation is denoted as y, the derivative is represented as $\frac{dy}{dx}$.

- The process of finding the derivative, $f'(x)$ or $\frac{dy}{dx}$, is referred to as **differentiation**.

- Derivatives may be evaluated from first principles or by using a set of rules.

- The derivative of a linear expression will result in a numeric value. From a graphical perspective, the gradient of a line is constant (it doesn't change) and hence the derivative will also be a constant value.



The gradient of **ANY** non linear relation is a function of x. In other words, the gradient of the tangent changes along a curve and is dependent upon the value of x. Therefore, the derivative of a non linear expression will result in an equation in terms of x.

Note:

- $\frac{dy}{dx}$ is an operation – it is not a quotient (it is not the same as dy ÷ dx).

- $\frac{d}{dx}(\ )$ reads as "the derivative of $(\ )$ with respect to x.

- The derivative may also be represented by the following notation: $D_x(f)$

Whenever you see the following words/phrase - gradient, gradient function, gradient of the tangent - automatically think of differentiation.

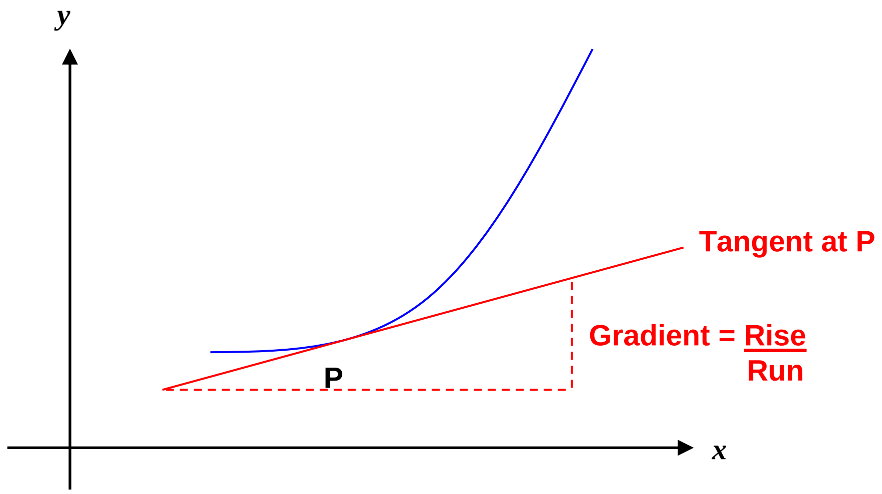
DERIVATIVES FROM FIRST PRINCIPLES

We can use limits to find derivatives using a process that is referred to as "differentiation using first principles". The derivative from first principles is obtained by applying the rule:

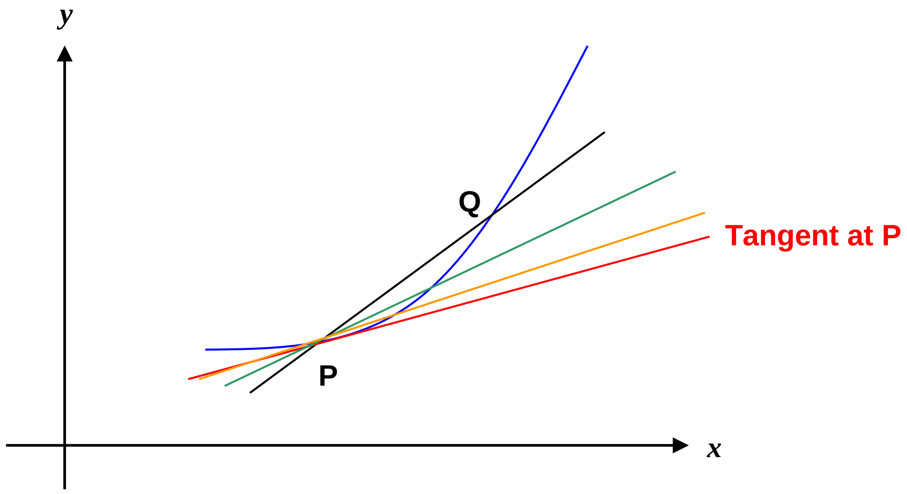
$$\text{Derivative} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\big(f(x)\big)$$

ORIGINS OF THE FIRST PRINCIPLES FORMULA

The gradient at a point P may be found by evaluating the gradient of the tangent to the curve at that point.



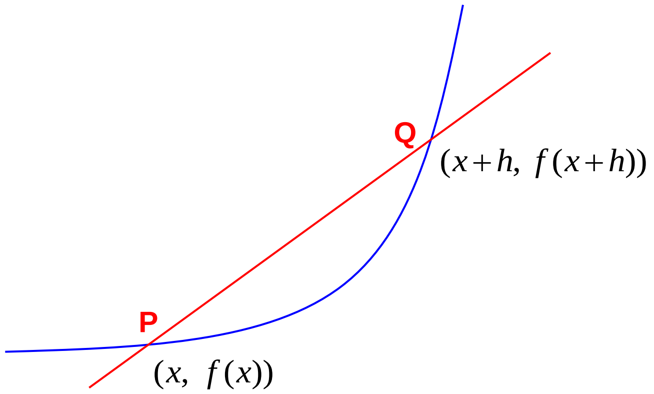
Consider any two points (P and Q) that lie on the same curve. As the distance between the two points approaches zero, the gradient of the line connecting the two points becomes a better approximate for the gradient of the tangent at P.



Let the x coordinate of point P equal x. The corresponding value of y will therefore be $f(x)$.

Let the distance between the x coordinates of the two points be represented as h.

The x coordinate of point Q will therefore equal $x+h$, and the corresponding value of y will be $f(x+h)$.



The gradient (m) of the line connecting the points P(x_1, y_2) and Q(x_1, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{change\,in\,y}{change\,in\,x} = \frac{rise}{run}$$

Therefore:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$m = \frac{f(x+h) - f(x)}{h}$$

As $h \to 0$, the gradient of the line connecting the two points becomes a better approximate for the gradient of the tangent at P. This is equivalent to finding the limit of the gradient function as $h \to 0$.

Gradient of a Tangent = Instantaneous Rate of Change =
Derivative = $\frac{dy}{dx} = \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h}$

An alternative notation for the limit theorem is:

$$\lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)]}{\delta x}$$

where δx is a small increment (change) in x.

FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

The derivative from first principles is obtained by applying the rule:

$$\text{Derivative} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} = f'(x) = \frac{dy}{dx}$$

Step 1: Write expressions for $f(x)$ and $f(x+h)$.

To obtain $f(x+h)$ from $f(x)$, replace x in the given equation with $(x+h)$.
For Example: If $f(x) = x^2 - 6x + 1$, then $f(x+h) = (x+h)^2 - 6(x+h) + 1$.

Step 2: Substitute the expressions for $f(x)$ and $f(x+h)$ into the rule

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

Step 3: Expand and collect like terms.
Step 4: Remove h as a common factor and simplify.
Step 5: Substitute $h = 0$.

To find the derivative at a specific value of x, substitute that value of x into the derived equation.

Note: The gradient at $x = 2$ is denoted as $f'(2)$.

QUESTION 10 – EXAM 1

Find from first principles, the derivative of $f\left(x\right)=3x^{2}+5x-1$.

Solution

Write the expressions for $f(x)$ and $f(x+h)$:

$$f\left(x\right)=3x^{2}+5x-1$$

$$f(x+h)=3(x+h)^{2}+5(x+h)-1$$
$$=3(x^{2}+2xh+h^{2})+5x+5h-1=3x^{2}+6xh+3h^{2}+5x+5h-1$$

Substitute the expressions for $f(x)$ and $f(x+h)$ into the limit theorem:

$$\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}=\lim_{h\to 0}\frac{[3x^{2}+6xh+3h^{2}+5x+5h-1]-[3x^{2}+5x-1]}{h}$$

Expand and collect like terms:

$$\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}=\lim_{h\to 0}\frac{3x^{2}+6xh+3h^{2}+5x+5h-1-3x^{2}-5x+1}{h}$$

$$=\lim_{h\to 0}\frac{6xh+3h^{2}+5h}{h}$$

Remove h as a common factor and simplify:

$$\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}=\lim_{h\to 0}\frac{h(6x+3h+5)}{h}=\lim_{h\to 0}(6x+3h+5)=6x+5 \quad ,h\neq 0$$

IMPORTANT NOTES

- The $\lim\limits_{h\to 0}$ notation was included in each step, until $h=0$ was substituted into the equation.
- You may check your answers by differentiating the given equation by rule. If the two answers are different, it is likely that an error has been made when expanding the brackets in the limit formula. Remember to multiply every term in the second set of brackets by negative one!

QUESTION 11 – EXAM 1

For the function with equation $f(x)=1-2x^3$:

(a) Find the gradient at any point x using first principles.

(b) Hence find the gradient of the tangent(s) at $x = 0$ and $x = 2$.

(c) Find the coordinates of the point on the curve at which the gradient of the tangent equals -6.

QUESTION 12 – EXAM 2

Show, using first principles, that $\lim_{x \to \infty} \left(\frac{x^2 - 3x + 2}{1 - x^2} \right) = -1$.

Solution

DIFFERENTIATING BY RULE

DERIVATIVES OF POLYNOMIAL AND RATIONAL FUNCTIONS

The derivative of an algebraic term is obtained by multiplying the coefficient by the power and then lowering the power by one.

If $y = ax^n$ **then** $\frac{dy}{dx} = anx^{n-1}$

This rule applies for all algebraic expressions of the form $y = ax^n$ providing that $n \neq 0$.

QUESTION 13 – EXAM 1

Find the derivative of $y = 3x^{-2}$.

Solution

Multiply the constant by the power, then lower the power by one:

$$\frac{dy}{dx} = 3 \times -2 \times x^{-2-1}$$
$$= -6x^{-3} = -\frac{6}{x^3}$$

This answer describes the gradient along the curve $y = 3x^{-2}$.

QUESTION 14 – EXAM 1

Find the derivative of $y = 2x^{3/4}$.

Solution

Multiply the constant by the power, then lower the power by one:

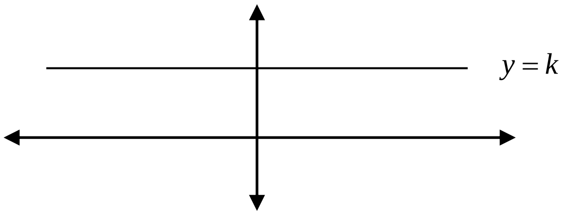
$$\frac{dy}{dx} = 2 \times \frac{3}{4} \times x^{3/4-1} = \frac{6}{4} \times x^{-1/4} = \frac{3}{2x^{1/4}}$$

This answer describes the gradient along the curve $y = 2x^{3/4}$.

THE DERIVATIVE OF A CONSTANT

If $y = k$ then $\frac{dy}{dx} = 0$

The derivative of a constant (a term that does not contain any variables) is equal to zero, as the graph of $y = k$ represents a straight line parallel to the X axis.



For Example:

(a) $y = 3$ $\quad \frac{dy}{dx} = 0$

(b) $y = 6c$ $\quad \frac{dy}{dx} = 0$

QUESTION 15 – EXAM 1

Find $\frac{dy}{dx}$ given that $y = -\frac{a^3 b}{c}$.

Solution

THE DERIVATIVE OF THE SUM OR DIFFERENCE OF A SERIES OF TERMS

The derivative of the sum and/or difference of terms is equal to the sum/difference of the derivatives of each individual term.

If $y = ax^2 + bx + c$ then $\frac{dy}{dx} = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx) + \frac{d}{dx}(c) = 2ax + b$

Note: $\frac{d}{dx}$ means "differentiate the given expression with respect to x".

QUESTION 16 – EXAM 1

Find the derivative of $y = 4 - 3x^3 + \frac{5x}{2}$.

Solution

Differentiate each term individually, then add/subtract the results:

$$y = 4 - 3x^3 + \frac{5x}{2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(4) - \frac{d}{dx}\left(3x^3\right) + \frac{d}{dx}\left(\frac{5x}{2}\right)$$

$$\frac{dy}{dx} = -9x^2 + \frac{5}{2}$$

QUESTION 17 – EXAM 1

Find $\frac{dy}{dx}$ given that $y = \frac{abx^3}{c} + 5x^{a+1} - (a+1)x^b$.

Solution

$$\frac{dy}{dx} = \frac{3abx^2}{c} + 5(a+1)x^a - (a+1)bx^{b-1}$$

FINDING DERIVATIVES – GENERAL APPROACH

METHOD:

Step 1: Rewrite all terms as powers on x.

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$(\sqrt[p]{x})^q = (x^{\frac{1}{p}})^q = x^{\frac{q}{p}}$$

Step 2: Bring terms involving x in the denominator (bottom of a fraction) to the top, by changing the sign on the power.

For example: $\frac{1}{x^2} = x^{-2}$ **Note:** $\frac{1}{6x^2} = \frac{x^{-2}}{6}$

Step 3: Simplify expressions so that terms are separated by addition and subtraction. Then differentiate each term individually. Alternatively, use algebraic techniques to reduce expressions to one term.

Step 4: Differentiate.

Step 5: Re-write the answer using positive powers. Bring terms with negative powers in the numerator (top of a fraction) to the bottom, by changing the sign on each power.

Step 6: State any restrictions on the values of x.

SIMPLIFYING EXPRESSIONS

Products:

- Expand simple products rather than applying the Product Rule.

- Apply index laws to convert products involving the same base into single terms.

  For example: $x^{m+1}.x^{1-2m} = x^{2-m}$

Quotients:

- Remove common factors and simplify.

- Factorise and eliminate terms by cancellation.

- If there is only one term in the denominator, write each term in the numerator over the denominator so that individual fractions are produced. Then simplify each term by cancellation.

  For example: $\frac{x^2+1}{x^2} = \frac{x^2}{x^2} + \frac{1}{x^2} = 1 + x^{-2}$ (Don't use the Quotient Rule).

- Simplify expressions using Index Laws.