VECTORS

This Topic Involves:

- *Addition and subtraction of vectors and their multiplication by a scalar, position vectors.*
- *Linear dependence and independence of a set of vectors and geometric interpretation.*
- *Magnitude of a vector, unit vector, the unit vectors* $\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k}$ *.*
- *Resolution of a vector into rectangular components.*
- *Scalar (dot) product of two vectors: Definition* $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}|.|\underset{\sim}{b}|\cos\theta$ *.*
- *Parallel and perpendicular vectors.*
- *Scalar product of two vectors and its use to find scalar and vector resolutes.*
- *Vector proofs of simple geometric results, for example:*

  The diagonals of a rhombus are perpendicular.
  The medians of a triangle are concurrent.
  The angle subtended by a diameter in a circle is a right angle.

VECTORS AND SCALARS

Quantities that may be specified by simply stating their magnitude are referred to as **scalar** quantities. For example:

- Distance
- Speed
- Mass

Quantities which need both magnitude and direction for their complete specification are known as **vector** quantities. For example:

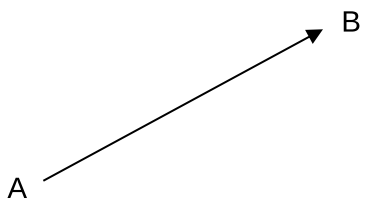
- Displacement 10 km NE
- Velocity 5 m/s , –5 m/s (These act in opposite directions)

Vectors are geometrically represented using a directed line segment. The length of the line segment represents the magnitude of the vector, while the orientation shows the direction of the vector.

NOTATION

Vectors are usually named using the small letters of the alphabet. These letters are written with a ~ (tilde) underneath. For example: $\underset{\sim}{b}$.

Vectors may also be named by stating the initial and terminal points (end points) using capital letters. For example, the below vector is represented as $\overrightarrow{AB}$.



The magnitude or modulus of the vector $\overrightarrow{AB}$ is denoted by $\left|\overrightarrow{AB}\right|$. In the same way, we may denote the magnitude of $\underset{\sim}{a}$ by $\left|\underset{\sim}{a}\right|$.

Vectors may be defined as being free or localised.

Localised vectors are characterised as having specific start and end points.

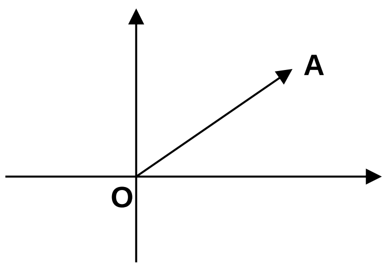
Free vectors are not localised i.e. they do not have specific start and end points in space.

Position Vectors:

When using Cartesian coordinate axes, it is often useful to refer to position vectors.

A position vector is a displacement vector whose starting point is the origin. Any point's position can be given in terms of a vector from the origin to that point.

For example:



$\overrightarrow{OA}$ is the position vector of A relative to O.

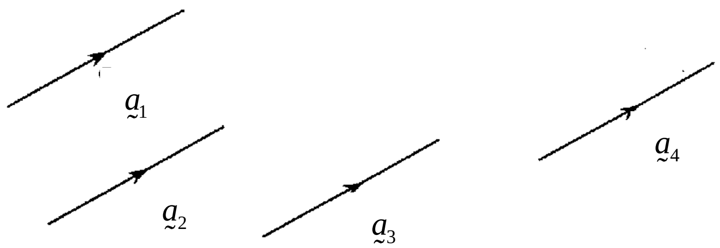
Note: When a question says that the position vectors are given by $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ this means that $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$

EQUALITY OF VECTORS

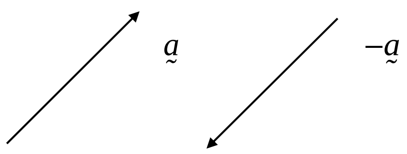
Two vectors are equal if they have the same magnitude and direction.

$\underset{\sim}{a} = \underset{\sim}{b}$ $\qquad\qquad$ $\underset{\sim}{a} \neq \underset{\sim}{b}$

Note: Vectors that are equal do not need to have the same starting point. This means that vector $\underset{\sim}{a}$ can be represented by infinitely many directed line segments, each with the same magnitude and direction, as illustrated below.



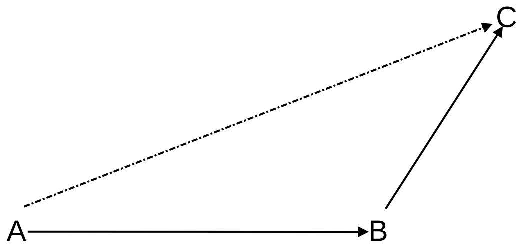
If $\underset{\sim}{a}$ is a vector then $-\underset{\sim}{a}$ is a vector with the same magnitude but acting in the opposite direction.



ADDITION OF VECTORS

When adding two vectors, you are finding the **resultant** vector.

For example:

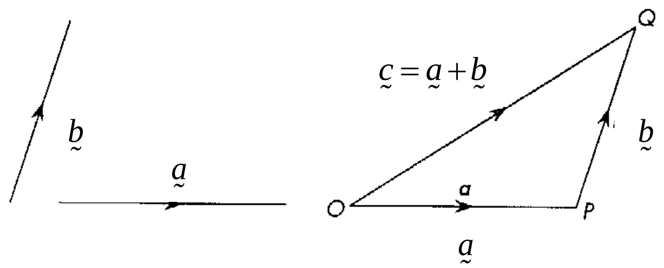


$\overrightarrow{AC}$ is the resultant vector.

Note: $\left|\overrightarrow{AC}\right| < \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$

The sum of two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, involves the combination not only of their magnitudes, but also of their directions.

To add two vectors, we use the "top to tail" method. This means that you draw the top of the first vector onto the tail of the second vector. The resultant is then drawn from the tail of the first vector to the top of the second vector.



Note:

- $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| < \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$
- If $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$ then $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$.
- Vectors obey the commutative law of addition i.e. $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$.
- Vectors obey the associative law of addition i.e. $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \underset{\sim}{c} = \underset{\sim}{a} + \left(\underset{\sim}{b} + \underset{\sim}{c}\right)$.

THE IDENTITY VECTOR

The identity vector is a vector which when added to any vector, leaves the vector unchanged.

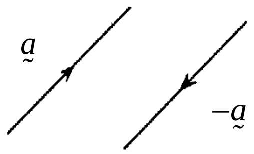
i.e. $\underset{\sim}{a}+\underset{\sim}{0}=\underset{\sim}{0}+\underset{\sim}{a}=\underset{\sim}{a}$

The zero vector has a zero magnitude and no particular direction.

THE INVERSE VECTOR

The inverse vector is one that when added to another vector, gives the zero vector.

For each vector ($\underset{\sim}{a}$), there exists a unique vector $-\underset{\sim}{a}$, such that $\underset{\sim}{a}+-\underset{\sim}{a}=-\underset{\sim}{a}+\underset{\sim}{a}=\underset{\sim}{0}$.

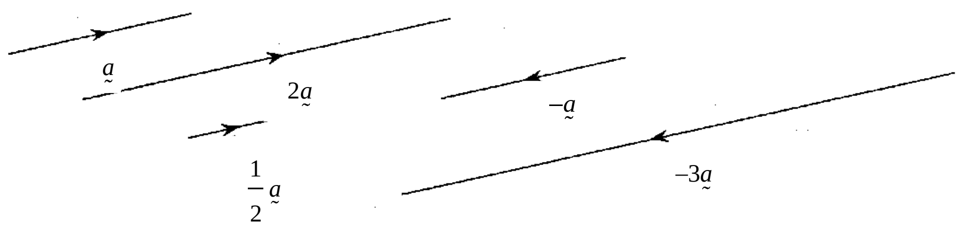


MULTIPLICATION OF A VECTOR BY A SCALAR

If $\underset{\sim}{a}$ is a vector and k is any real number, then $k\underset{\sim}{a}$ is a vector with a magnitude k times as big as $\underset{\sim}{a}$, acting parallel to $\underset{\sim}{a}$.

- If $k>0$ then $k\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$.
- If $k=0$ then $k\underset{\sim}{a}=\underset{\sim}{0}$.
- If $k<0$ then $k\underset{\sim}{a}$ has the opposite direction to $\underset{\sim}{a}$

For example:



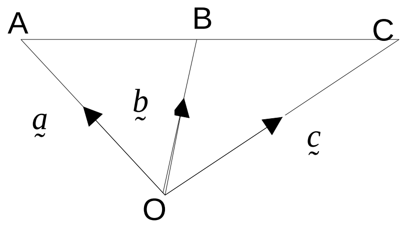
Note: Multiplication of a vector by a scalar results in a vector.
$k\underset{\sim}{a} = \underset{\sim}{a}k$

PARALLEL AND COLLINEAR VECTORS

Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are said to be parallel if $\underset{\sim}{a} = k\underset{\sim}{b}$ where k is a scalar.

If two parallel vectors have a point in common, they are said to be COLLINEAR.

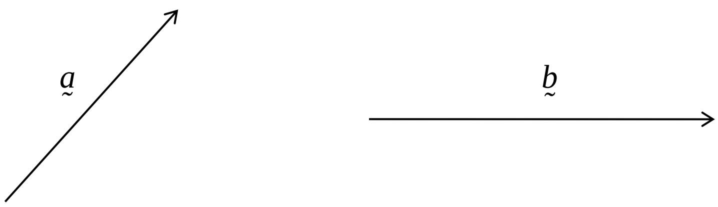
For example: If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then the points A, B and C are collinear as shown in the diagram below.



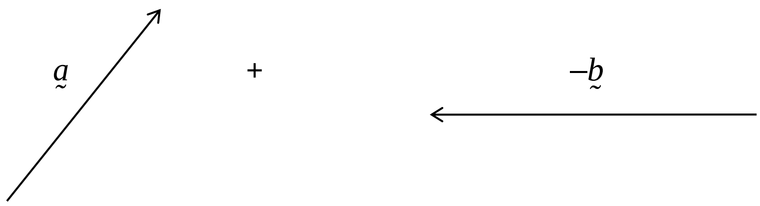
SUBTRACTION OF VECTORS

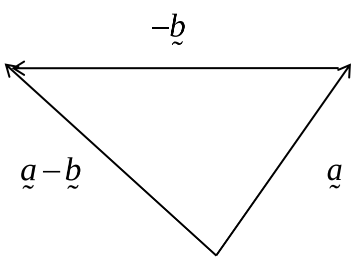
Subtraction is equivalent to the addition of a negative vector. Simply reverse the direction of the negative vector and add the two vectors "top to tail".

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$$



$\therefore\ \underset{\sim}{a} - \underset{\sim}{b} =$

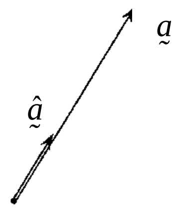




(See Questions 11, 12 and 13)

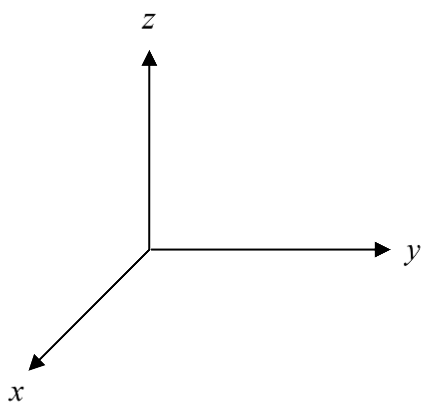
UNIT VECTORS

A **unit vector** is a vector whose magnitude is **one**. These vectors are used as direction indicators, and are denoted by placing the symbol ^ on top of the vector name. For example, the unit vector in the direction of $\underset{\sim}{a}$ is denoted as $\underset{\sim}{\hat{a}}$.



VECTORS IN 3-DIMENSIONAL SPACE

To describe the position of a point in space, three coordinates are required. If we use the Cartesian system, only two of the coordinates of a point can be represented (x and y). The third Cartesian coordinate (z) is obtained by introducing a third axis, OZ, which lies perpendicular to the X and Y axis.



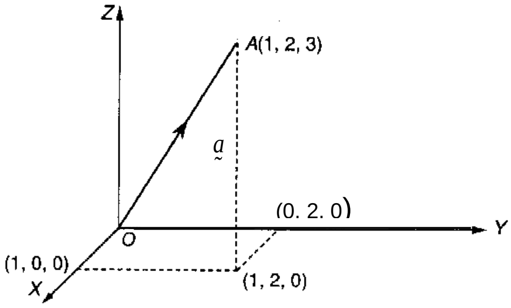
- A unit vector in the X-direction is defined as $\underset{\sim}{i}$.
- A unit vector in the Y-direction is defined as $\underset{\sim}{j}$.
- A unit vector in the Z-direction is defined as $\underset{\sim}{k}$.

We can use this three-dimensional Cartesian system to describe the position of any point.

Note:

- To represent a point in vector notation we need to discuss its position in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$.
- The expression $3\underset{\sim}{i}$ represents a vector of magnitude 3 units parallel to the X-axis.
- The point P with co-ordinates (2,1,3) is written as $\overrightarrow{OP} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

For example: $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ is located 1 unit across in the X – direction.
2 units across in the Y – direction.
3 units up in the Z – direction.



POSITION VECTORS

The vector $\overrightarrow{OP}$ is the position vector of P relative of O.

If a question states the position vector for A is $2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ this really means that $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

THE MAGNITUDE (SIZE) OF A VECTOR

The magnitude (size) of a vector in three-dimensional space is determined by using an extension of Pythagoras' Theorem.

If $\underset{\sim}{P} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ then the magnitude of $\underset{\sim}{P}$ is given by $|\underset{\sim}{P}| = \sqrt{x^2 + y^2 + z^2}$.

Simply:

1. Square the coefficients of $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$.
2. Add together.
3. Square root the result.

For example:

If $\underset{\sim}{P} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ the magnitude of $\underset{\sim}{P}$ is equal to:

$$|\underset{\sim}{P}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

If $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ then $|\underset{\sim}{a}| = \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}$.

CREATING UNIT VECTORS

To create a unit vector out of any vector, we simply divide the vector by its magnitude.

Thus, the unit vector of a vector $\underset{\sim}{a}$, in the direction of $\underset{\sim}{a}$ is $\underset{\sim}{a} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

For example: The magnitude of $\underset{\sim}{P}$ is 3. Therefore, the unit vector will be:

$$\underset{\sim}{P} = \frac{1}{3}(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$$

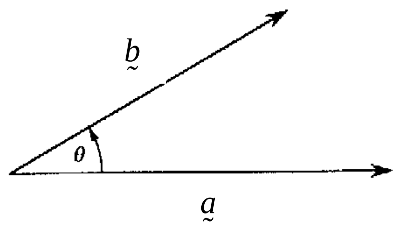
This vector lies parallel to vector $\underset{\sim}{P}$ and has a magnitude of 1.

(See Questions 14 and 15)

THE SCALAR PRODUCT OF TWO VECTORS

The scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is the product of their magnitudes $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$, multiplied by the cosine of the angle between the two vectors.

$$\boxed{\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}|.|\underset{\sim}{b}|\cos\theta}$$



This rule is known as the scalar product or dot product, and is used to find the angle between two vectors.

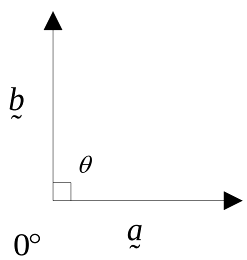
Note: The answer is scalar.
θ represents the angle between two vectors positioned tail to tail.

SPECIAL CASES

1. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then $\theta = 90°$

$\therefore \cos\theta = 0$

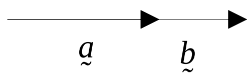
$\therefore \underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|.0 = 0$



2. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel (same direction) then $\theta = 0°$

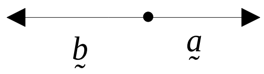
$\therefore \cos\theta = 1$

$\therefore \underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|.1 = |\underset{\sim}{a}||\underset{\sim}{b}|$



3. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel (opposite direction) then θ = 180°

$\therefore \cos\theta = -1$



$\therefore \ \underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|.-1 = -|\underset{\sim}{a}||\underset{\sim}{b}|$

> If $\underset{\sim}{a}.\underset{\sim}{b} = 0$ then $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$
>
> If $\underset{\sim}{a}.\underset{\sim}{b} = \pm|\underset{\sim}{a}|.|\underset{\sim}{b}|$ then $\underset{\sim}{a}$ must be parallel to $\underset{\sim}{b}$

SCALAR PRODUCT IN COMPONENT FORM

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$

$$\therefore \quad \underset{\sim}{a}.\underset{\sim}{b} = x_1x_2 + y_1y_2 + z_1z_2$$

Note: $x_1\underset{\sim}{i}.y_2\underset{\sim}{j} = x_1y_2\underset{\sim}{i}.\underset{\sim}{j} = 0$ since $\underset{\sim}{i}.\underset{\sim}{j} = 0$ ($\underset{\sim}{i}$ is perpendicular to $\underset{\sim}{j}$)

(See Question 16)

ANGLES BETWEEN VECTORS

If $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}|.|\underset{\sim}{b}|\cos\theta$ then $\cos\theta = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}|.|\underset{\sim}{b}|}$

For vectors written in component form:

As $\underset{\sim}{a}.\underset{\sim}{b} = x_1x_2 + y_1y_2 + z_1z_2$ and $|\underset{\sim}{a}| = \sqrt{x^2 + y^2 + z^2}$ then

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$$

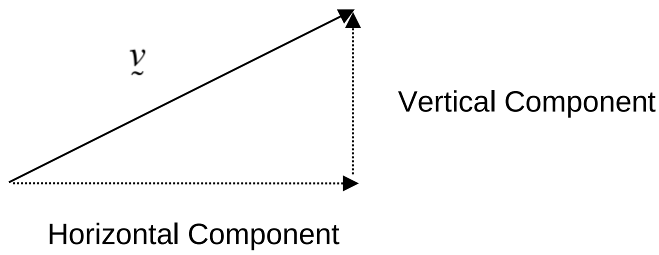
(See Question 17)

RESOLUTION OF VECTORS

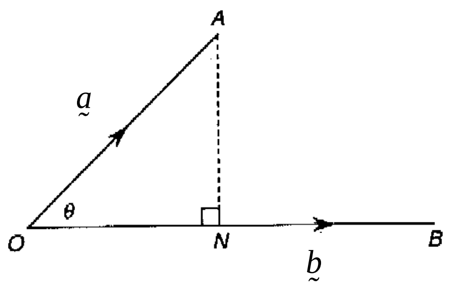
Just as two vectors can be added to give one resultant vector, one vector ($\underset{\sim}{v}$) can be expressed as the sum of two other vectors. This process is called resolving a vector into its separate component parts.

These component parts are usually resolved in two perpendicular directions which would give vector $\underset{\sim}{v}$ as a resultant vector.

Vectors are usually resolved as the sum of two perpendicular vectors – the horizontal and vertical components. This process is known as resolving the vector into rectangular components.



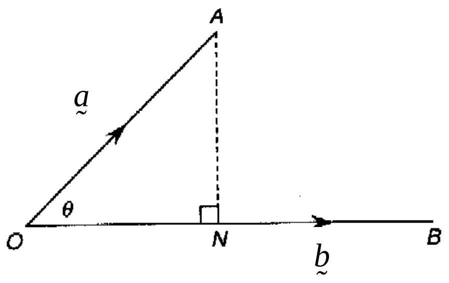
By breaking down a vector into its components, we are able to analyse the effect of the vector in one particular direction.

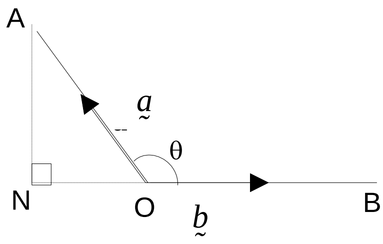


The direction of one of the components will generally be stated.

SCALAR RESOLUTES

$\theta < 90°$ $\theta > 90°$





The scalar resolute is simply the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ (the amount of shadow that $\overrightarrow{OA}$ casts on $\overrightarrow{OB}$). In simpler terms, the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is equal to the magnitude of $\overrightarrow{ON}$.

The projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b} = \left|\overrightarrow{ON}\right|$

As $\cos\theta = \dfrac{ADJ}{HYP}$**:** $\cos\theta = \dfrac{\left|\overrightarrow{ON}\right|}{\left|\overrightarrow{OA}\right|}$ $\therefore \left|\overrightarrow{ON}\right| = \left|\overrightarrow{OA}\right|\cos\theta$

Sub $\overrightarrow{OA} = \left|\underset{\sim}{a}\right|$ $\therefore \left|\overrightarrow{ON}\right| = \left|\underset{\sim}{a}\right|\cos\theta$

As $\cos\theta = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|.\left|\underset{\sim}{b}\right|}$ $\therefore \left|\overrightarrow{ON}\right| = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$

Sub $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$ $\therefore \left|\overrightarrow{ON}\right| = \underset{\sim}{a}.\hat{\underset{\sim}{b}}$

This expression gives us the scalar resolute of $\underset{\sim}{a}$ upon $\underset{\sim}{b}$ or projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ or $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. It is a scalar quantity (a number) and has no direction.

Note:

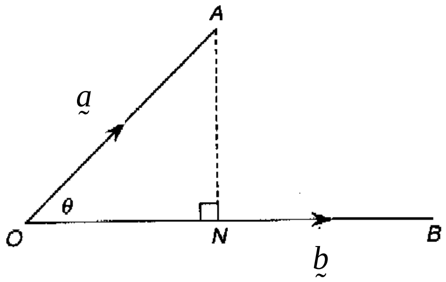
- If θ is an acute angle, then $\left|\overrightarrow{ON}\right|$ is a positive value.
- If θ is an obtuse angle, then $\left|\overrightarrow{ON}\right|$ is a negative value.

The **scalar resolute** of $\underset{\sim}{a}$ **parallel** to $\underset{\sim}{b}$ or $\underset{\sim}{a}$ **upon** $\underset{\sim}{b}$ or the **projection** of $\underset{\sim}{a}$ **onto** $\underset{\sim}{b}$ or $\underset{\sim}{a}$ **in the direction of** $\underset{\sim}{b}$ is $\underset{\sim}{a}.\hat{\underset{\sim}{b}}$ where $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$, therefore $\underset{\sim}{a}.\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$.

Using Pythagoras' Theorem the **scalar resolute** of $\underset{\sim}{a}$ **perpendicular** to $\underset{\sim}{b}$ $= \sqrt{\left|\underset{\sim}{a}\right|^2 - \left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)^2}$.

VECTOR RESOLUTES

Vector resolutes describe the magnitude and direction of vector components.



The vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ is equal to the magnitude and direction of $\overrightarrow{ON}$.
(The horizontal component.)

$\therefore\ \overrightarrow{ON} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\ \hat{\underset{\sim}{b}}$ ← This gives direction

Scalar component (this gives the magnitude).

The **<u>vector resolute</u>** of $\underset{\sim}{a}$ **parallel** to $\underset{\sim}{b}$: $\underset{\sim}{a}_{\parallel} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \left(\dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\underset{\sim}{b}.\underset{\sim}{b}}\right)\underset{\sim}{b}$

The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is defined as the magnitude and direction of $\overrightarrow{NA}$ (the vertical component).

The **<u>vector resolute</u>** of $\underset{\sim}{a}$ **perpendicular** to $\underset{\sim}{b}$: $\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel}$

Note:

The **<u>vector resolute</u>** of $\underset{\sim}{b}$ **parallel** to $\underset{\sim}{a}$ is defined as: $\underset{\sim}{b}_{\parallel} = \left(\underset{\sim}{b} \cdot \underset{\sim}{a}\right)\underset{\sim}{a} = \left(\dfrac{\underset{\sim}{b}.\underset{\sim}{a}}{\underset{\sim}{a}.\underset{\sim}{a}}\right)\underset{\sim}{a} = \left(\dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\underset{\sim}{a}.\underset{\sim}{a}}\right)\underset{\sim}{a}$

The **<u>vector resolute</u>** of $\underset{\sim}{b}$ **perpendicular** to $\underset{\sim}{a}$ is defined as: $\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - \underset{\sim}{b}_{\parallel}$

(See Questions 18, 19 and 20)

GEOMETRIC PROOFS USING VECTORS

The geometry theorems listed in this section should be familiar to you, and you should be able to apply them.

THEOREMS REGARDING THE SUMS OF ANGLES

- The sum of the interior angles of a triangle is 180°.
- An exterior angle of a triangle is equal to the sum of the two opposite interior angles.
- The sum of the interior angles of a quadrilateral is 360°.
- The sum of the interior angles of a polygon with n sides is $(n - 2) \times 180°$.
- The sum of the exterior angles (taken clockwise or anti-clockwise) of a convex polygon is 360°.

THEOREMS REGARDING TRIANGLES AND PARALLELOGRAMS

- Opposite angles of a parallelogram are equal.
- Opposite sides of a parallelogram are equal in length.
- The base angles of an isosceles triangle are equal.
- The line joining the vertex to the midpoint of the base of an isosceles triangle is perpendicular to the base.
- The perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.
- The line segment joining the mid-points of two sides of a triangle is parallel to the third side and half its length.
- The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.

THEOREMS REGARDING CIRCLE GEOMETRY

The following theorems regarding angles in a circle should also be committed to memory:

- The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference.
- The angle in a semicircle is a right angle.
- Angles in the same segment of a circle are equal.
- The sum of the opposite angles of a cyclic quadrilateral is 180°.
- An exterior angle of a cyclic quadrilateral and the interior opposite angle are equal.
- The two (common) tangents to a circle from an exterior point are equal in length.
- A tangent to a circle is perpendicular to the radius to the point of contact.
- The angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.

VECTOR PROOFS

In order to prove geometrical theorems using vectors we need to recall some basic vector facts.

(i) If $\underset{\sim}{u} = k\underset{\sim}{v}$ then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel vectors.

(ii) If two parallel vectors have a point in common then they are collinear.

(iii) If two vectors are perpendicular then their dot product is zero.

(See Questions 21 and 22)

QUESTION 11

In a trapezium $RSTU$, RS is parallel to and twice the length of UT. If $\overrightarrow{UT} = \underset{\sim}{p}$ and $\overrightarrow{TS} = \underset{\sim}{q}$,

find $\overrightarrow{UR}$, in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

Solution

QUESTION 12

$OABC$ is a parallelogram in which $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$ and M is the midpoint of AB.

(a) Express $\overrightarrow{OM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(b) Express $\overrightarrow{CM}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

(c) If P is a point on OB such that $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Hence, show that $\underset{\sim}{p} = \frac{1}{3}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)$.

(d) Use the fact that $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{b}$ to express $\underset{\sim}{p}$ in terms $\underset{\sim}{b}$ of only. What is the ratio of the lengths $OP : PB$?

QUESTION 13

The angle between 2 vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 50°, the vectors have magnitudes of 9 and 14 respectively. Find the resultant vector $2\underset{\sim}{a}+\underset{\sim}{b}$ correct to 2 decimal places.

Solution

QUESTION 14

Find a vector of magnitude 9 in the opposite direction to $-\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$.

Solution

Let $\underset{\sim}{r}=-\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$.

$$\left|\underset{\sim}{r}\right|=\sqrt{(-1)^2+(2)^2+(2)^2}=\sqrt{9}=3$$

Let $\underset{\sim}{v}$ be the required vector. Then $\left|\underset{\sim}{v}\right|=3\left|\underset{\sim}{r}\right|$.

In order to be in the opposite direction $\underset{\sim}{v}=-3\underset{\sim}{r}$

Therefore $\underset{\sim}{v}=-3\left(-\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}\right)=3\underset{\sim}{i}-6\underset{\sim}{j}-6\underset{\sim}{k}$.

QUESTION 15

Given $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = -3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -2\underset{\sim}{j} + \underset{\sim}{k}$, find:

(a) A unit vector in the direction of $\underset{\sim}{c}$.

(b) The magnitude of $\underset{\sim}{a} + 2\underset{\sim}{b}$.

(c) A unit vector parallel to $\underset{\sim}{a} + 2\underset{\sim}{b}$.

Solution

QUESTION 16

If $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, find the value of p for which

$\left(p^2\underset{\sim}{a} + p\underset{\sim}{b}\right).\underset{\sim}{c} = \underset{\sim}{a}.\underset{\sim}{b}$.

Solution

QUESTION 17

If $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ find:

(a) The magnitude of $\underset{\sim}{u}$.

(b) The angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Give answer correct to 2 decimal places.

Solution

(a) $|\underset{\sim}{u}| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$

(b) $\cos\theta = \dfrac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = -\dfrac{4}{9}$

$\theta = 116^o 23'$

Note: Since the ratio is negative, the angle between the two vectors is obtuse.

QUESTION 18

If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are defined as follows $\underset{\sim}{u} = 4\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$,

(a) Find a unit vector in the direction of $\underset{\sim}{v}$.

(b) Express $\underset{\sim}{u}$ as the sum of two components – one in the direction of $\underset{\sim}{v}$, the other perpendicular to $\underset{\sim}{v}$.

Solution

(a) $\hat{\underset{\sim}{v}} = \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$ $\quad \therefore \hat{\underset{\sim}{v}} = \dfrac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(b) $\left(\underset{\sim}{u}.\hat{\underset{\sim}{v}}\right)\hat{\underset{\sim}{v}}$ is the vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$.

However, it is easier to use $\dfrac{(\underset{\sim}{u}.\underset{\sim}{v})\underset{\sim}{v}}{|\underset{\sim}{v}|^2} = \dfrac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

Perpendicular component:

$$\underset{\sim}{u} - \left(\frac{2}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}\right) = 4\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k} - \left(\frac{2}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}\right)$$

$$= \frac{10}{3}\underset{\sim}{i} + \frac{5}{3}\underset{\sim}{j} - \frac{10}{3}\underset{\sim}{k}$$

$$= \frac{5}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

QUESTION 19

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$. The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $3x\underset{\sim}{i}-2\underset{\sim}{j}+4\left(x-2\right)\underset{\sim}{k}$.

(a) Show that $x=4$.

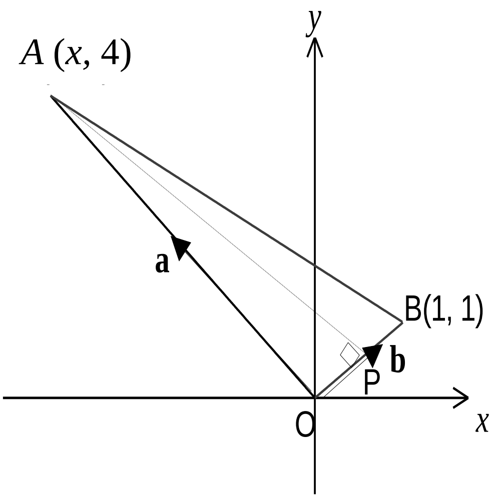
(b) Hence, find $\underset{\sim}{a}$.

Solution

QUESTION 20

Point $O(0, 0)$, $A(x, 4)$ and $B(1, 1)$, where $x \in R^-$ form the vertices of a triangle as shown in the diagram below.

The position vectors $\overrightarrow{OA} = \underset{\sim}{a} = x\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\overrightarrow{OB} = \underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j}$ are indicated. The angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is θ where $\cos\theta = \dfrac{1}{5\sqrt{2}}$. AP is an altitude of triangle OAB.



(a) Show that $x = -3$ or $-\dfrac{16}{3}$.

(b) Using $x = -3$, find the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

(c) Hence, find the length of the altitude AP.