CORE: RECURSION AND FINANCIAL MODELLING

It should be noted that this material was new to the Core of the Further Maths course in 2016. Some of the coursework was previously contained in the modules Business Maths and Number Patterns, but there is some difference in emphasis and some material that is new this year or has been removed from the old module material.

In recent years an ongoing problem in the Business Maths module has been students not giving enough detail in monetary answers by rounding to the nearest dollar or the nearest five cents.

Currency in Australia has two decimal places and while cash transactions are rounded to the nearest five cents, electronic transactions are commonly carried out to the nearest cent. Some financial transactions, such as petrol prices and share prices, are calculated using fractions of cents. As per instructions in the 2016 examination, unless otherwise stated in a question, monetary answers should always be given correctly rounded to the nearest cent.

INVESTMENTS AND LOANS

An investment is where an amount of money is put into a bank or other investment account, with the intention that it will gain in value. The gain in value can be from interest paid on the amount investment or a combination of interest and additional payments.

A loan is where a bank or other financial institution gives money to a person, usually for a specific purpose such as buying a house or a car. The money is paid back usually with regular payments, but in addition to the amount borrowed, interest is paid on the outstanding balance.

The amount borrowed or invested at the start is called the principal.

Interest that is applied to a loan or an investment can be calculated as simple interest or compound interest.

Simple interest is where interest is only applied to the original amount of the investment or loan. The interest remains the same during the time the account is held as it is always calculated on the starting amount.

Compound interest is where the interest is applied to the current balance of the account. The interest paid each period on an investment increases as the balance increases. The interest paid on a loan each period decreases as the amount owed decreases.

Most interest rates are quoted as “per annum” rates or pa. This means “per year”.

RECURSION

The Further Maths course requires that most loans and investments be considered using recurrence relations. The advantage of using recurrence relations is that a better understanding of the growth of investments or decay of loans can be gained.

Recursion or recurrence means to repetitively apply the rule to each value to obtain the next value.

Each recurrence relationship must have:

- A starting point and
- A rule that allows successive terms in the sequence to be generated.

For Further Maths, the starting point will be V_0 or A_0 or B_0 etc. This will usually be the principal.

Each term is V_1, V_2, V_3 etc. The general term for the n^{th} term is V_n, which means the value V after n repeats of the rule. Other letters than V may be used.

This means that if we want the value of an investment after 8 years then we look for V_8 etc.

We also use V_{n+1} to mean the next term in the sequence after V_n. The next term again would be V_{n+2} and the next term V_{n+3} etc.

Therefore the rule is written so that V_{n+1} is obtained by a repeated mathematical operation on V_n.

EXAMPLE
For the sequence of amounts in an account, $400, $480, $560, $640,… write the rule for the sequence.

Solution

The starting term is $400, so V_0= 400.

The next term is obtained by adding $80 to the previous term, so the rule would be:

First term is $400   next term = current term + 80

The recurrence relation would be $V_0 = 400 \quad V_{n+1} = V_n + 80$

EXAMPLE
For the sequence of amounts in an account, \$50, \$100, \$200, \$400,… write the rule for the sequence.

Solution

The starting term is \$50, so V_0= 50.

The next term is obtained by multiplying the previous term by 2, so the rule would be:

First term is \$50   next term = 2 × current term

The recurrence relation would be $V_0 = 50 \quad V_{n+1} = 2\,V_n$

Note: Be very careful about how you write recurrence rules. You should ensure that you write the subscripts clearly as such. There is a very big difference between V_{n+1} and $V_n + 1$!

QUESTION 1
For the sequence of amounts in an account, \$300, \$320, \$340, \$360,… write a recurrence relation in terms of A_n and A_{n+1} for this sequence.

Solution

QUESTION 2
For the sequence of amounts in an account \$20, \$60, \$180, \$540,… write a recurrence relation in terms of B_n and B_{n+1} for this sequence.

Solution

INVESTMENTS

SIMPLE INTEREST

For simple interest the recurrence relation for the balance of an investment is given by:

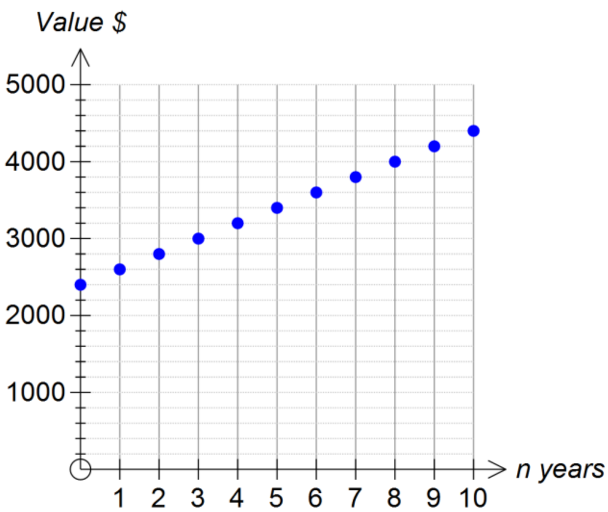
$$starting\ value = principal \qquad next\ value = current\ value + interest\ added$$

This can be written as a mathematical recurrence relation: $V_0 = P \quad V_{n+1} = V_n + D$

where V_n is the value of the investment after n time periods and D is the interest added based on the original value.

D can be calculated using $D = \frac{r}{100} \times V_0$ where r is the per annum interest rate.

The graph of a simple interest investment is a straight line (or often it is drawn as a series of points in a straight line) where V_0 is the y-intercept and the gradient is the value of D.



This graph shows a simple interest investment because the points would form a straight line. The y-intercept is $2400, so the principal is $2400. The gradient is $200, so the amount added each year is $200. The recurrence relation would be $V_0 = 2400 \quad V_{n+1} = V_n + 200$.

Simple interest is an example of an arithmetic sequence. An arithmetic sequence has a common difference between its terms.

EXAMPLE
What is the total amount in a $12 000 investment after 4 years at 5.2% per annum simple interest?

Solution

To write the recurrence relationship we need to know the actual value added to the account each year:

$$Amount\ added = \frac{5.2}{100} \times 12000 = \$624$$

The recurrence relationship can now be written:

$V_0 = P \qquad V_{n+1} = V_n + D$

$V_0 = 12000 \quad V_{n+1} = V_n + 624$

The amount after each year can now be determined recursively using technology or manually:

$V_1 = V_0 + 624 = 12000 + 624 = 12\,624$

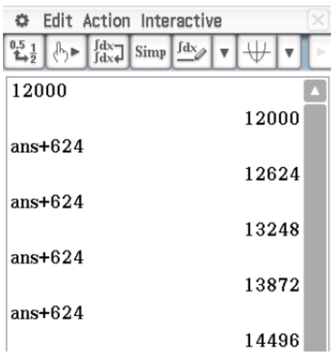
$V_2 = V_1 + 624 = 12624 + 624 = 13248$

$V_3 = V_2 + 624 = 13248 + 624 = 13872$

$V_4 = V_3 + 624 = 13872 + 624 = 14496$

The study design specifically states that you must be able to generate values from "first principles" as above for up to the first five values.

Casio ClassPad

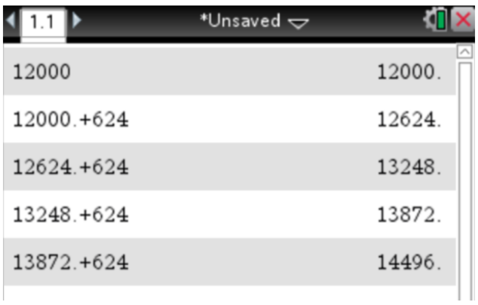


The initial value is entered and enter/EXE pressed to make the start the output

→ the recursion is set up using the "Ans" function

→ using Ans + 624 and repeating by pressing enter or EXE will give subsequent values

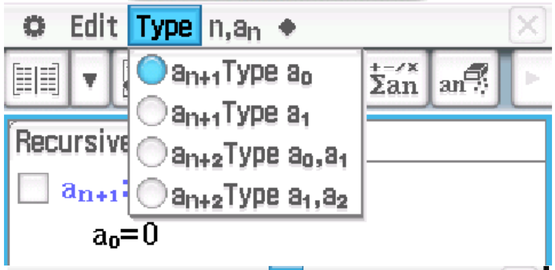
TI-Nspire



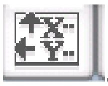
SEQUENCING ON THE CASIO CLASSPAD

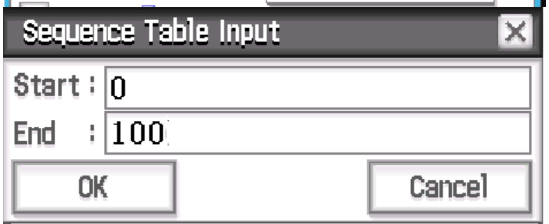
The Casio ClassPad has a specific sequencing menu.

The menu should be on the recursive tab and the type should be as shown



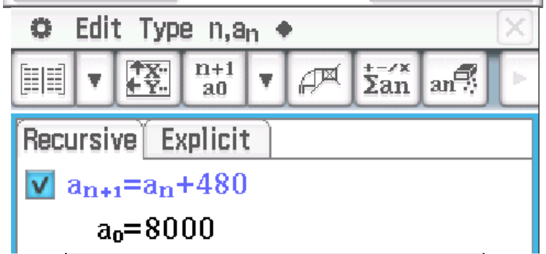
The summing display should be “off” as shown

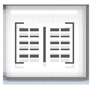
The table range should be set using .



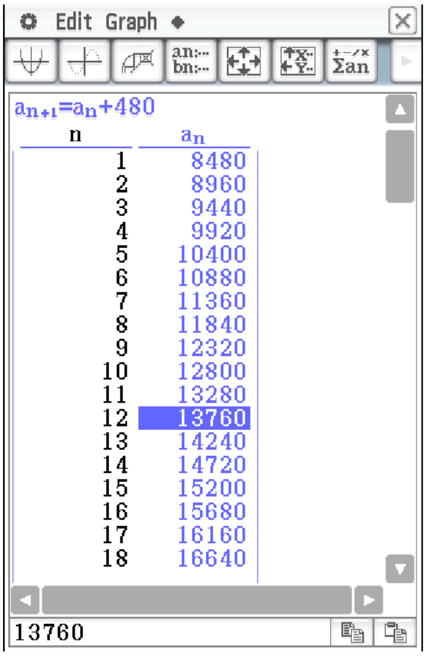
The range must start at 0 and up to 100 will cover most sequences required.

Set up sequence as shown and tick to select.



 will produce the sequence in the lower screen.

The resize button can be used or you can scroll down to see the value.



Note: Be careful that you select values and then read them from the bottom of the screen to ensure that full detail is given. This is particularly important when there are decimals as a rounded value is often displayed in the table.

QUESTION 3
An investment of $6000 accrues 4.4% per annum simple interest. The amount of interest added each year is:

A $26.40
B $2640
C $1363
D $2.64
E $264

QUESTION 4
The value C_8 of the sequence $C_0 = 4800 \quad C_{n+1} = C_n + 240$ is:

A 6240
B 6480
C 6720
D 6960
E 7200

QUESTION 5
The simple interest rate for the sequence of values $C_0 = 4800 \quad C_{n+1} = C_n + 240$ is:

A 2.4%
B 4.2%
C 0.05%
D 6%
E 5%

QUESTION 6
A recurrence relationship that could model the sequence of values after n years for an investment of $5600 at 3.7% per annum simple interest is:

A $V_0 = 5600 \quad V_n = V_{n+1} + 207.20$
B $V_1 = 5600 \quad V_{n+1} = V_n + 207.20$
C $V_0 = 5600 \quad V_{n+1} = V_n + 207.20$
D $V_0 = 5600 \quad V_{n+1} = V_n + 370$
E $V_0 = 5600 \quad V_{n+1} = 1.037 \times V_n$

Sometimes you don't want to scroll through every term to get to your answer. Another formula for simple interest is:

$V_n = V_0 + D \times n$ where $D = \frac{r}{100} \times V_0$

This could also be written as $V_n = V_0 + \frac{r}{100} \times V_0 \times n$.

This rule allows you to go directly to a term in a sequence or by using a solver to calculate an unknown rate or principal.

EXAMPLE
What is the total amount in a $15 000 investment after 12 years at 4.5% per annum simple interest?

Solution

After 12 years the required term is V_{12}. The principal or V_0 is $15 000. The value of r is 4.5%.

$$V_n = V_0 + \frac{r}{100} \times V_0 \times n$$
$$V_{12} = 15000 + \frac{4.5}{100} \times 15000 \times 12 = \$23\,100$$

EXAMPLE
What interest rate would result in $14 976 interest from an investment of $26 000 over an eight year period?

Solution

$$V_0 = 14976 + 26000$$
$$V_n = V_0 + \frac{r}{100} \times V_0 \times n$$
$$V_8 = (14976 + 26000) = 26000 + \frac{r}{100} \times 26000 \times 8$$

Using your solver:

$$\text{solve}\left(14976+26000=26000+\frac{x}{100}\cdot 26000\cdot 8, x\right)$$
$$\{x=7.2\}$$

The rate is 7.2%.

QUESTION 7
What is the total amount in an $8000 investment after 7 years at 3.7% per annum simple interest?

Solution

QUESTION 8
What initial investment would be required to have a total balance of $10 000 after 3 years at a simple interest rate of 6.5% per annum?

A $512.82
B $8 368.20
C $2 994.01
D $51 282.05
E $8 278.49

COMPOUND INTEREST

Compound interest is much more common for both loans and investments than simple interest. It is calculated on the current balance of the account rather than the original balance. Any interest payments or additional payments made during the course of the investment or loan are included in the interest calculation.

The interest can be calculated annually (once per year), biannually (twice a year/every 6 months), quarterly (4 times a year/every 3 months), monthly, weekly or even daily.

The interest rate per time period is the per annum rate divided by the number of compounds per year, so an interest rate of 6% per annum is equivalent to:

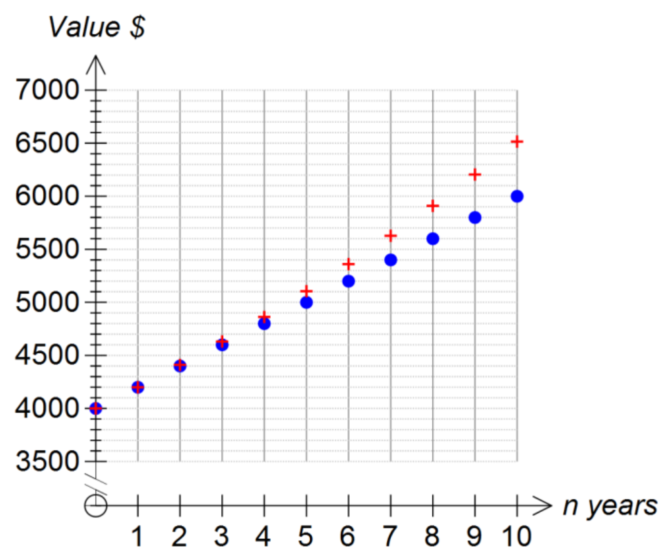
- $6 \div 2 = 3\%$ every half year (six months)
- $6 \div 4 = 1.5\%$ every quarter
- $6 \div 12 = 0.5\%$ every month
- $6 \div 52 = \frac{6}{52}\%$ every six months
- $6 \div 365 = \frac{6}{365}\%$ every six months

Note: If the rate does not divide to an **<u>exact</u>** decimal, then a fraction should be used.

You must also be careful NEVER to round any value used in sequencing as you will compound your error for every time period calculated.

Generally, for an investment at the same interest rate per annum, the more often that the investment compounds, the more money is made from the investment.

A compound interest investment or loan graph is a curve rather than a straight line (given by simple interest). A compound interest investment of $4000 over a 10 year period at 5% per annum compounding annually is shown below along with the simple interest investment also at 5% per annum. Note that in the long run the compound interest investment is better because interest is being calculated on the increasing balance.



Compound interest must also be understood as a recurrence relationship. Compound interest is an example of a geometric sequence where there is a common ratio (multiplier) between terms.

For a compound interest investment or loan where there are no deposits or payments into the account, the balance after n months will be given by the recurrence relationship:

$$starting\ value = principal \qquad next\ value = percentage\ increase\ \times current\ value$$

$$V_0 = P \qquad V_{n+1} = (1 + \frac{r}{100})V_n$$

The value $(1 + \frac{r}{100})$ is often written as a single number, for example:

$\left(1 + \frac{6}{100}\right) = 1.06$ $\quad$ $\left(1 + \frac{4.3}{100}\right) = 1.043$ $\quad$ $\left(1 + \frac{0.7}{100}\right) = 1.007$

If you have rate such as $\frac{6}{365}\%$ then this should be left as $\left(1 + \frac{\frac{6}{365}}{100}\right) or\ (1 + \frac{6}{36500})$ so that you don't introduce an ongoing error.

EXAMPLE
A compound interest investment of $9000 accrues interest at 3.6% per annum compounding monthly.

(i) Write a recurrence relation in terms of S_{n+1} and S_n that would model this investment.

(ii) Determine the amount in the account after 3 months correct to the nearest cent from first principles.

Solution

The interest rate per month is $3.6 \div 12 = 0.3$.

The value $(1 + \frac{0.3}{100})$ is written as 1.003. The principal is $9000. The rule for the sequence is:

$S_0 = 9000 \qquad S_{n+1} = (1 + \frac{0.3}{100}) \times S_n$ OR $S_0 = 9000 \qquad S_{n+1} = 1.003\, S_n$

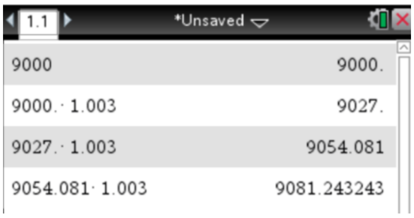
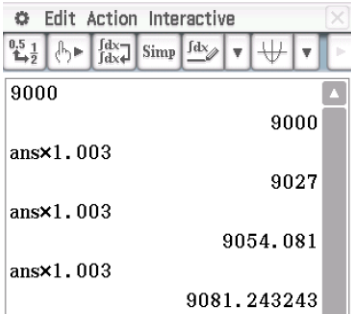
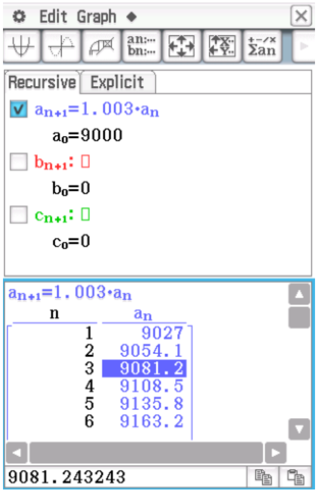
The terms in the sequence can be determined using the recurrence relation from first principles:

$S_1 = 1.003 \times S_0 = 1.030 \times 9000 = \$9027$

$S_2 = 1.003 \times S_1 = 1.003 \times 9027 = \$9054.081 \approx \$9054.08$

$S_3 = 1.003 \times S_2 = 1.003 \times 9054.081 = \$9081.243243 \approx \$9081.24$

Alternatively the calculator can be used:

TI-Nspire	**Casio ClassPad (main)**	**Casio ClassPad (sequence)**
		

Questions 9, 10 and 11 refer to the following information:

The recurrence relation $H_0 = 12000, \quad H_{n+1} = 1.056\, H_n$ models the value of an investment after n years.

QUESTION 9
The annual interest rate for this investment is:

A 56%
B 5.6%
C 1.056%
D 0.56%
E 1.56%

QUESTION 10
The value of the investment after 9 years is closest to:

A $18 048.00
B $17 572.30
C $18 556.35
D $19 596.51
E $20 692.86

QUESTION 11
The amount of interest earned during the 5th year was closest to:

A $835.65
B $672
C $882.45
D $791.34
E $748.37

QUESTION 12
An investment of $15 000 accrues 6.4% interest compounding quarterly. A recurrence relation that could model the value of the investment after *n* quarters would be:

A $V_0 = 15\,000 \quad V_{n+1} = V_n + 960$
B $V_0 = 15\,000 \quad V_n = 1.064\,V_{n+1}$
C $V_0 = 15\,000 \quad V_n = 1.064V_{n-1}$
D $V_0 = 15\,000 \quad V_{n+1} = 1.016\,V_n$
E $V_0 = 15\,000 \quad V_{n+1} = 1.16\,V_n$

Sometimes instead of using a recurrence relation it is easier to go directly to a particular future value or you may need to solve for the rate or principal. The rule to find a particular term V_n is:

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Notice that this is the same as:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

You may have used this rule in the past in the second version, but the first version uses the terminology of recurrence relations.

EXAMPLE
Determine the value of a $7000 investment at 6% interest compounding monthly after 3 years correct to the nearest cent.

Solution

The interest rate per month is $\frac{6}{12} = 0.5\%$ per month and there are 3 × 12 = 36 months (time periods) in 3 years.

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$
$$V_{36} = \left(1 + \frac{0.5}{100}\right)^{36} \times 7000$$

$$V_{36} = 1.005^{36} \times 7000 \approx \$8376.76$$

QUESTION 13
Sarah invests $15 000 at 9% per annum compound interest, compounding monthly for a period of 5 years. Which of the following formulae represents the amount that is in Sarah's account after this period?

A $15\,000 \times (1 + \frac{9}{100})^5$

B $15\,000 \times \left(1 + \frac{0.75}{100}\right)^5$

C $15\,000 \times (1 + \frac{0.75}{100})^{60}$

D $15\,000 \times \left(1 + \frac{9}{100}\right)^{60}$

E $15\,000 \times (1 + \frac{5}{100})^9$