SYSTEMATIC COUNTING METHODS

PERMUTATIONS (ORDER MATTERS)

Consider trying to form a queue of five people **A**manda, **B**rian, **C**lare, **D**aniel and **E**mily.
In such a question the order clearly matters since being at the head of a queue is usually preferable to being last in line (except perhaps if you are awaiting an injection or the like!)

Such an **ordered** arrangement of a number of items without replacing or repeating any of them is known as a **permutation**. The arrangement of the items can be regarded as a series of successive events and is best handled using the box notation below.

Consider the letters A, B, C, D and E. We have to arrange them in as many ways as we can using all four but each letter can only be used once.

Put any of the five at the head of the queue.	Put any one of the remaining four in second position.	Put any one of the remaining three here.	Put any one of the remaining two here.	You only have one left so put it here!!

So the number of arrangements – permutations – is: $5 \times 4 \times 3 \times 2 \times 1 = 120$

We can use the notation 5! to denote this product of the five successively decreasing integers being multiplied together. This is called a **factorial**. Hence we can talk about:

5! ("5 factorial") is $5 \times 4 \times 3 \times 2 \times 1 = 120$

Similarly $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

So start with the number before the exclamation mark symbol and multiply all the numbers in decreasing order until you get to 1. You can also use the factorial button on your calculator, usually marked n!

If we do not want to use all five letters – say just two letters – and arrange them in order then we can

Put any one of the five in this position.	Put any one of the remaining four here.

Now the number of arrangements – permutations – is: $5 \times 4 = 20$

For this we can use the notation $^5P_2$ – which means from a group of five objects, select two and determine the number of ways those two can be arranged.

So $^5P_2 = 5 \times 4 = 20$. This can be expressed using factorials as:

$$\frac{5!}{(5-2)!} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}}$$

Again there is a button on your calculator, usually marked ^nP_r.
The number of permutations of n objects taken r at a time is denoted ^nP_r where

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)(n-3)........(n-r+1)$$

This statement can and will be very useful for a number of problems.

SPECIAL CASES

$$\boldsymbol{r = 1}$$

Note that if we only take **ONE** object from a set of n, $^nP_1 = n$ and also that, by definition, $0! = 1$ (i.e. how many ways can you arrange no objects – must be 1.)

$$\boldsymbol{r = n}$$

Also we have the special case with which we started, that of the number of arrangements of taking **ALL** *n* objects in an ordered arrangement is

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

EXAMPLE 8
How many different 4 digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6 and 7 if there are no repetitions and:

(a) No restrictions?
(b) The number must be bigger than 5000?

Solution

(a) Here we are arranging 7 objects taken 4 at a time – so its time for boxes:

Put any one of the seven digits in this position.	Put any one of the remaining six here.	Put any one of the remaining five here.	Put any one of the remaining four here.

$\therefore$ The number of ordered arrangements is $7 \times 6 \times 5 \times 4 = 840$
Or using a permutation, $^7P_4 = \frac{7!}{(7-4)!} = 840$.

This second part is a great example of the general principle in counting questions that we should first deal with any parts which involve a restriction.

General Principle: Deal with restrictions first

(b) For a number being bigger that 5000, the first digit can only be 5, 6 or 7. This restricts the first digit choice to one of only three not seven!

Put any one of the three possible digits in this position.	We have six digits left so any one of the remaining six here.	Put any one of the remaining five here.	Put any one of the remaining four here.

Hence the number of ordered arrangements is $3 \times 6 \times 5 \times 4 = 360$ or using a permutation,

$$3 \times {}^{6}P_{3} = 360$$

This question could have been developed further by asking a variation on the question which requires even more care as follows:

(c) How many numbers less than 2000 can be formed using these digits?

Important note – there is **NO** mention of the *number of digits* in these numbers (so there may not necessarily be four digits – as we had previously). This is a type of question where we have no choice but to consider each case separately and brings us to another useful principle.

Useful Principle: Avoid using cases unless there is no obvious alternative

Number of single digit numbers: _________

Number of two digit numbers: $7 \times 6 = 42$

Number of three digit numbers: $7 \times 6 \times 5 = 210$

Number of four digit numbers: __ $\times$ __ $\times$ __ $\times$ __ = ____, since _________________

Total:

This next example involves a question where there are sub-groups within the arrangement. As you will see the general principle to use here is to first arrange all the pieces by treating those grouped together as a single entity and then to consider the number of rearrangements within the sub-group. If there are only two objects A and B, they can be ordered as either AB or BA, if there are three A, B and C then there are 3!=6 orderings within the group etc.

General Principle: First order the groups, then consider the internal ordering of each group.

EXAMPLE 9

In how many ways can our 5 friends, **A**manda, **B**rian, **C**lare, **D**aniel and **E**mily:

(a) Stand in a queue without restrictions on their ordering?

(b) Stand in a queue if **D**aniel is at the front and **A**manda is at the rear?

(c) Stand in a queue if **C**lare and **D**aniel must stand together?

(d) If **A**manda and **B**rian are together and **D**aniel and **E**mily are together?

(e) Stand in a queue if **C**lare and **D**aniel must stand apart? (The fickleness of young love!)

Solution

(a) $5! = 120$ (or ${}^5P_5$)

(b) D at front and A at rear with both in fixed positions. So only arrangements involve the other 3 which gives $1 \times 1 \times 3! = 6$.

(c) Consider Clare and Daniel together as a single entity, (the happy couple) and hence arrange 4 groups – so $4! = 24$.

Then consider internal ordering of the couple, Clare first then Daniel or vice versa, so 2 or if you prefer 2!

Finally multiply these two factors.
Total = $4! \times 2! = 48$

(d) We have now have only 3 groups: Clare on her own and the two couples Amanda with Brian and Daniel with Emily. Order these first so 3! ways.

Now consider internal orderings of each couple, 2 for Amanda and Brian and a further 2 for Daniel and his new partner.

Now multiply these factors
Total = $3! \times 2! \times 2! = 24$ ways.

(e) Tricky!!

Often people start trying to solve these type of questions using cases such as

Case 1: one person separating the fighting couple

Case 2: two people sandwiched between the warring parties

Case 3: the pair are on either end as far apart as possible

While such an approach can sometimes be used successfully it is **MUCH EASIER** to use a **COMPLEMENTARY APPROACH**. By this I mean find how many arrangements there are of the **OPPOSITE** restriction and subtract these from the **UNRESTRICTED** total.

Hence the number of arrangements with Clare and Daniel apart is actually the unrestricted total from part (a) MINUS the number where they are next to each other found in part (c) = 120 - 48 = 72

This approach works since Clare and Daniel are either together or they are apart, these are complementary events.

Useful Principle: Sometimes a question is best answered using the complementary approach

QUESTION 31

Three Mathematics books, six French books and four Music books are to be arranged on a shelf. All the books are different. In how many ways can the books be arranged if:

(a) There are no restrictions?

(b) The books from each subject are to be together?

QUESTION 32
Three boys and three girls are to be arranged in a row. How many arrangements are possible if:

(a) There are no restrictions.

(b) The boys and girls must alternate.

(c) A girl must be in the first position and a boy in the last position.

(d) 2 particular girls must not be together (note no other restrictions mentioned).

(e) An extra girl is added to the group and again the boys and girls must alternate.

QUESTION 33

Five people are to travel together to a concert. The car they are to travel in seats 2 people in the front and 3 in the back.

(a) If there are 2 people who can be the driver, find the number of ways the 5 people can be seated.

(b) In how many ways can the people be arranged if one of the passengers (not a driver), must have a window seat?

QUESTION 34

Eight rowers are to be arranged in a boat with four on each side.
How many arrangements are possible if:

(a) There are no restrictions?

(b) One rower (James) must sit on the front right seat and another rower (Alex) must sit on the front left seat?

(c) James and Alex must occupy the front two seats?

(d) Two rowers must sit on the left side of the boat and one rower must sit on the right side.

QUESTION 35

How many numbers can be formed with the digits 0, 1, 5, 6, 7, 9 if there are no repetitions (numbers cannot start with 0) and:

(a) All digits must be used?

(b) The number must be bigger than 600 000?

(c) All digits are used and the number is odd?

ARRANGEMENTS IN A CIRCLE

When objects are arranged in a circle, the total number of arrangements is reduced due to the symmetry on rotation. The arrangement of four people in a line is easy as we have already seen. However, with a circle the arrangement ABCD is exactly the same as the arrangement BCDA, cycling all the letters one place forward, – all we (as the observer or the waiter) have to do is to walk one place around the circle and we have exactly the same arrangement as the first.

A circle does not have a beginning so it is necessary to place one person into their seat to start the process and then arrange the other people as though they were being ordered in a line. So for 4 people, sit one down and arrange the others (in 3! ways) relative to the starting person.

In general, the number of ways of arranging *n* objects in a circle (without restriction) is:

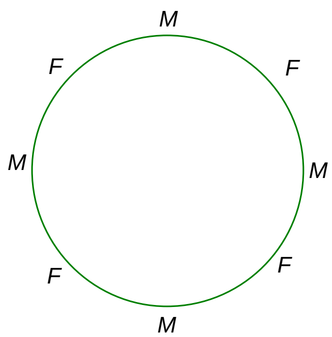
$$(n-1)!$$

There are *(n-1)!* ways of arranging *n* items in a circle

QUESTION 36
In how many ways can 4 men and 4 women be seated around a circular table if:

(a) There are no restrictions?

(b) The men and women are seated alternately?



(c) Two particular men are not to sit together? (Hint: Use complementary approach)

QUESTION 37
Five boys are to sit around a circular table. In how many ways can this be done if:

(a) There are no restrictions?

(b) Two of the boys are not allowed to sit together?

ARRANGEMENTS WITH REPETITIONS

The total number of possible arrangements is also reduced when there are identical objects being considered. The most common example is a word which has repeated letters. Here any particular arrangement looks to be the same if (say) two letter Es are interchanged. So what we have to do is to remove the effect of the number of ways the repetition can be made.

EXAMPLE 9
In how many ways can the letters of the word MATHEMATICS be arranged in a row with no restrictions?

Solution

There are 11 letters in the word MATHEMATICS, 2 each of M, A and T and one each of the remainder.

The number of ways to arrange 11 letters is 11! = 39 916 800.

But now, by exchanging the two Ms, we get MATHEMATICS – looks familiar?

As there are 2! ways of arranging the two Ms and every arrangement of the 11 letters has that duplication, we must divide the total number of arrangements by 2!.

The same logic must apply to the two Ts and to the two As.

$\therefore$ The number ways the letters can be arranged is $\frac{11!}{2!2!2!} = 4989600$, a mere five million!!

In summary:

If n objects of which p are alike of one type and q are alike of another and so on, are to be arranged in a row, then the number of arrangements is given by:

$$\frac{n!}{p!q!......}$$

For every set of *p* repetitions the number of arrangements is reduced by a factor of *p!*

QUESTION 38

Assume your favourite word is CALCULUS.

(a) How many different words can be formed from the letters of CALCULUS? (A word does not have to be a valid word in the English language simply an arrangement of letters.)

(b) How many of these arrangements begin with C?

(c) How many of these words end with S?

COMBINATIONS

(ORDER UNIMPORTANT)

A combination is the selection of objects from a large group but where order of selection (i.e. arrangement) is not important. In such a case, we might want to select two students from a class to represent the class on the SRC. What is important is that two particular students are selected – NOT in which order those two students are selected. So we want Lily and Tom, for example, and it does not matter which is chosen first.

We think of this as:

(a) How can we select two students from a class of 20 – ${}^{20}P_2$?

(b) How many ways could those two students be arranged? 2!

Hence the number of ways to select the two students in any order is:

$$\frac{{}^{20}\mathbf{P}_2}{2!} = 190$$

In general, the number of ways to select *r* objects from *n* where order of selection is not important is:

$${}^{n}\mathbf{C}_r = \frac{{}^{n}\mathbf{P}_r}{r!}$$

We use the symbol C to represent <u>combinations</u>. The number before the C and above is the total pool from which we make a selection. The number after and below the C is the number we a selecting without order being important.

EXAMPLE 10
The number of ways to select 5 people from a group of 10 is ${}^{10}C_5 = 252$.

Use the C button on your calculator to obtain the answer. Remember however what the C stands for. Here it means:

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5}$$

The expression ${}^{n}C_r$ can be extended to:

$${}^{n}C_r = \frac{n!}{r!(n-r)!}$$

You should prove that expression.

QUESTION 39
In how many ways can a committee of 5 people be selected from 6 men and 8 women if:

(a) There are no restrictions?

(b) There must be 3 women and 2 men on the committee?

(c) A particular man and woman are not allowed to serve on the same committee?

(d) There must be at least one woman on the committee?

QUESTION 40
A team of eleven players is to be selected from a soccer team consisting of 18 players, where 3 are goalkeepers, 8 are forwards and 7 are defenders.

How many ways can this be done if:

(a) There are no restrictions?

(b) The team is to be made up of a goalkeeper, 4 defenders, 3 midfielders (who can be chosen from any of the forwards and defenders) and 3 forward players?

(c) The team must have **at least** 2 forwards? (Hint: The phrase ‘at least’ can often imply cases or suggest using a complementary approach to avoid cases.)

QUESTION 41
A registration number plate is to consist of 2 letters followed by 3 digits. Registration plates starting with the letter Z or the arrangement “CD”, are not allowed. How many number plates are possible?

Solution

QUESTION 42
Six boys and three girls are to be seated in a room where the chairs are set up in two rows of 6. In how many ways can this be done if:

(a) The first row must be fully occupied?

(b) The three girls must sit together?

QUESTION 43

(a) How many ‘words’ can be made from the letters of the word PERMUTATIONS if all letters are to be used?

(b) In how many words will the vowels be together in their alphabetical order (a, e, i, o, u)?

QUESTION 44 2011 HSC Q 2(c)
Alex’s playlist consists of 40 different songs that can be arranged in any order.

(a) How many arrangements are there for the 40 songs?

(b) Alex decides that she wants to play her three favourite songs first in any order.

How many arrangements of the 40 songs are now possible?

QUESTION 45 (HSC 2014 Multiple Choice Q8)
In how many ways can 6 people from a group of 15 people be chosen and then arranged in a circle?

A $\frac{14!}{8!}$

B $\frac{14!}{8!6}$

C $\frac{15!}{9!}$

D $\frac{15!}{9!6}$

QUESTION 46 (HSC 2012 Multiple Choice Q5)
How many arrangements of the letters of the word OLYMPIC are possible if the C and the L are to be together in any order?

A 5 !

B 6 !

C 2x5 !

D 2x6 !

QUESTION 47 (HSC 2013 Multiple Choice Q7)
A family of eight is seated randomly around a circular table. What is the probability that the two youngest members of the family sit together?

A $\frac{6!2!}{7!}$

B $\frac{6!}{7!2!}$

C $\frac{6!2!}{8!}$

D $\frac{6!}{8!2!}$

QUESTION 48 (HSC 1993 Q4c)
Five travellers arrive at a town where there are five hotels.

(a) How many different accommodation arrangements are there if there are no restrictions on where the travellers stay?

(b) How many different accommodation arrangements are there if each traveller stays in a different hotel?

(c) Suppose two of the travellers are husband and wife and must go to the same hotel. How many different accommodation arrangements are there if the other three can go to any of the other hotels?