Piecewise functions

Nick Egbert

MA 158 Lesson 9

Warm-up. Find and simplify the difference quotient

$$\frac{f(4+h)-f(4)}{h},$$

where $f(x) = 13x^2 + 6$.

Solution.

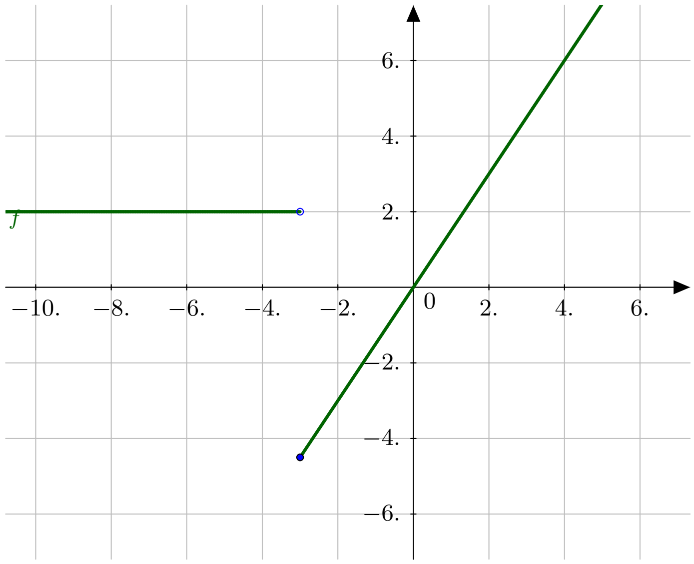
$$\begin{aligned}\frac{13(4+h)^2+6-\left(13(4^2)+6\right)}{h} &= \frac{13(16+8h+h^2)+6-13\cdot 16-6}{h}\\ &= \frac{13\cdot 16+13\cdot 8h+13h^2+6-13\cdot 16-6}{h}\\ &= \frac{\overbrace{13\cdot 8}^{104}h}{h}+\frac{13h^2}{h} = 104+13h\end{aligned}$$

□

Overview

This lesson is mainly concerned with finding domains of piecewise functions, and coming up with piecewise functions for application problems. Let's first look at the graph of f below. In case it is not clear in the picture, there is a hollow circle at $(-3, 2)$ and a filled-in circle at $(-3, -4)$. We can break up $f(x)$ into two functions $f_1(x)$, $f_2(x)$, and then we can then ask some questions:

1. $f_1(x) = \frac{3}{2}x$ for what values of x?
2. $f_2(x) = 2$ for what values of x?
3. What is the range of f?



Graph of f

Solution. For 1., looking at the graph, we see that $x \geq -3$ is the domain for $\frac{3}{2}x$. We include -3 since the circle is filled in. For 2., the domain is $x < -3$. In interval notation, these would be $(-3, \infty)$ and $(-\infty, -3)$, respectively. For 3., place your pencil on the graph horizontally. If your pencil would eventually hit some part of the graph, then that value is in the range. We can certainly do this going up to ∞ since the graph goes up to the right forever. And we can keep going down all the way until $y = -4.5$. So the range is $[-4.5, \infty)$. □

Remark. Remember that we always read the values off the graph from left to right, from bottom to top. Additionally, we always write intervals from smallest value to largest value.

Drawing piecewise functions

Example 1. Sketch the graph and find the following.

$$f(x) = \begin{cases} 2x + 14, & \text{if } x \leq -3 \\ 16 - 2x, & \text{if } x > -3 \end{cases}$$

(a) Domain of f

(b) Range of f

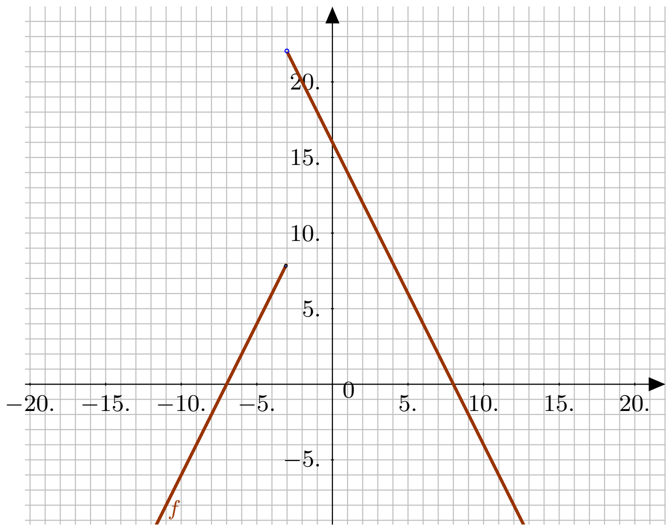
(c) $f(-12)$

(d) $f(-3)$

(e) $f(10)$

(f) $f(15)$

Solution. Here is the graph below.



(a) Using the vertical line test, the only potential problem is $x = -3$, but $f(-3)$ is defined, so the domain is $(-\infty, \infty)$.

(b) Using the horizontal line test, we can go down to $-\infty$, but only up to 22. So the range of f is $(-\infty, 22)$.

(c) $f(-12) = -10$

(d) $f(-3) = 8$

(e) $f(10) = -4$

(f) $f(15) = -14$ □

Remark. To find $f(a)$, look at how f is defined and find which domain a is in. For example, -3 belongs to $x \le -3$, so we need to use $2x + 14$ since that corresponds to $x \le -3$.

On graphing piecewise functions

To graph a piecewise function, it is a good idea to follow these steps.

1. Look at the inequalities first. Draw a dotted vertical line for each of these values.
2. Marking *lightly*, graph all the functions which are given for f.
3. Looking back at the inequalities, darken in the functions between the vertical lines for which they are valid.

More examples

Example 2. Sketch the graph of h and find the following.

$$h(x) = \begin{cases} x^2 + 5, & \text{if } -5 < x < 0 \\ 5, & \text{if } 0 \leq x \leq 3 \\ 9 - x, & \text{if } 3 < x \leq 10 \end{cases}$$

(a) Domain of h

(b) Range of h

(c) $h(-5)$

(d) $h(10)$

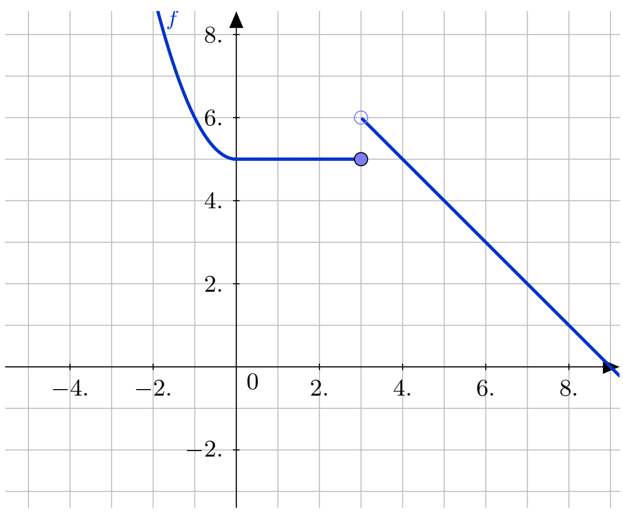
(e) $h(0)$

(f) $h(3)$

(g) $h(-10)$

(h) $h(4)$

Solution. Below is the graph for f.



(a) We need to be careful here. Unlike the previous examples, this graph doesn't go off to $\pm\infty$. Looking at the first inequality in the definition of h, the smallest x can be is -5, but not including -5. And the largest x can be is 10, looking at the last inequality. So the domain of h is $(-5, 10]$.

(b) The lowest point is when $x = 10$, and the y-value here is -1 (plug x into the last equation.) The highest point is when $x = -5$, and this has a y-value of $(-5)^2 + 5 = 30$. But since -5 is not included in the domain, 30 is not included in the range. So the range of h is $(30, -1]$.

(c) Since -5 was not included in the domain, $h(-5)$ does not exist.

(d) When $x = 10$, we are in the 3$^{\text{rd}}$ line for h. So $h(10) = -1$.

(e) When $x = 0$, we are in the 2$^{\text{nd}}$ line, and here $h = 5$, so $h(0) = 5$.

(f) For the same reasoning in (e), $h(3) = 5$.

(g) For the same reasoning in (c), $h(-10)$ does not exist.

(h) Again, we are in the 3$^{\text{rd}}$ line, so $h(4) = 9 - 4 = 5$. □

Example 3. Same instructions as before.

$$f(x) = \begin{cases} -5, & \text{if } 2 < x \leq 3 \\ -4, & \text{if } 3 < x < 4 \\ -3, & \text{if } 4 \leq x \leq 5 \\ -2, & \text{if } 5 < x \leq 6 \\ -1, & \text{if } 6 < x < 7 \end{cases}$$

(a) Domain of f

(b) Range of f

(c) $f(2)$

(d) $f(3)$

(e) $f(7)$

(f) $f(3.18)$

(g) $f(\sqrt{20.14})$

Solution. The graph of f is just horizontal lines at the y-values for each of the intervals specified. Make sure that you have a filled in circle on the inequalities $\leq, \geq$ and an open circle on the inequalities $<, >$.

(a) If you look closely at the inequality signs, you will see that every x is covered between 2 and 7, but 2 and 7 are not included in the domain of f, so the domain is $(2, 7)$.

(b) Since we have just these 5 horizontal lines, there are only 5 y-values, so the range of f is $\{-5\} \cup \{-4\} \, \{-3\} \cup \{-2\} \cup \{-1\}$. Note that we use the curly braces whenever we have a single point instead of an interval.

(c) $f(2)$ does not exist

(d) $f(3) = -5$ since $2 < 3 \leq 3$

(e) $f(7)$ does not exist

(f) $f(3.18) = -4$ since $3 < 3.18 < 4$

(g) $f(\sqrt{20.14}) = -3$ since $\sqrt{20.14} \approx 4.5$ and $4 \leq 4.5 \leq 5$. □

Example 4. A cellphone company offers two plans. Plan A costs \$40 a month for the first 150 minutes, and \$0.27 for each additional minute. Plan B costs \$60 a month for the first 200 minutes and \$0.2 for each additional minute.

(a) Find a piecewise function $C_A(t)$ that determines the total cost C_A of Plan A per month in dollars using t minutes.

(b) Find a piecewise function $C_B(t)$ that determines the total cost C_B of Plan B per month in dollars using t minutes.

(c) Rounding to the nearest minute, when does Plan B become less expensive than Plan A?

Solution. (a) Starting with Plan A, for $0 \leq t \leq 150$, the cost is a constant \$40, so on this interval $C_A(t) = 40$. Now for $t > 150$, the rate is .27/min, so .27 is our slope. But what is the equation of this line? Well, we know that when we use 150 minutes, the plan still costs \$40, so $(150, 40)$ is a point on the graph, and it should start sloping up at exactly this point. Now that we have slope and a point, we can use point slope form to find the second part of the piecewise function.

$$\begin{aligned} y &= .27(x - 150) + 40 \\ &= .27x - 40.5 + 40 \\ &= .27x - .5. \end{aligned}$$

Putting these two things together, we see

$$C_A(t) = \begin{cases} 40, & \text{if } 0 \leq t \leq 150 \\ .27t - .5, & \text{if } t > 150 \end{cases}.$$

(b) Similarly for B, for $0 \leq t \leq 200$, $C_B(t) = 60$. And for $t > 200$, we have a slope of .2. Now a point on this line is $(200, 60)$. So to find the equation for the second line, we again use point slope form:

$$\begin{aligned} y &= .2(t - 200) + 60 \\ &= .2t - 40 + 60 \\ &= .2t + 20. \end{aligned}$$

Now putting these together, we get

$$C_B(t) = \begin{cases} 60, & \text{if } 0 \le t \le 200 \\ .2t + 20, & \text{if } t > 200 \end{cases}.$$

(c) To find out when B becomes less expensive, it is easiest to find out when the plans are equal. To do this, we need to be careful since these are piecewise-defined functions. It should be clear that $C_A(t) < C_B(t)$ for $t \le 150$. When $t = 200$, then $C_B(200) = 60$ and $C_A(200) = .27(200) - .5 = 54 - .5 = 53.5 < 60$. So the equal each other for some $t > 200$.

To see when they are equal we set $C_A(t) = C_B(t)$ for $t > 200$

$$\begin{aligned} .27t - .5 &= .2t + 20 \\ .07t &= 20.5 \\ t = 20.5/.07 \approx 292.86 \approx 293. \end{aligned}$$

So when $t > 293$, Plan B will cost less than Plan A. □

Example 5. A certain county has a tax code, where 10% tax is paid on all income up to the first \$10,000, a 15% tax is paid for any income over \$10,000 and up to \$15,000, and a tax rate of 25% is paid on all income over \$25,000. Find a piecewise function to calculate the total tax $T(x)$ on an income of x dollars.

Solution. For $x \le 10,000$ the rate is .1, so $T(x) = .1x$ on this interval. For $10,000 < x \le 25,000$, the rate is .15. Similar to the previous example, we need to find a point. When $x = 10,000$, then $y = .1 \cdot 10,000 = 1,000$. Now using point-slope form,

$$\begin{aligned} T(x) &= .15(x - 10,000) + 1,000 && (1) \\ &= .15x - 1500 + 1000 && (2) \\ &= .15x - 500. && (3) \end{aligned}$$

Similarly for the rate of .25, we get $T(x) = .25x - 3000$ when $x > 25,000$. Putting these together,

$$T(x) = \begin{cases} .1x, & \text{if } x \le 10,000 \\ .15x, & \text{if } 10,000 < x \le 25,000 \\ .25x, & \text{if } x > 25,000 \end{cases}.$$

□

Example 6. A car rental agency has two different rental plans. Plan A costs \$100 per day and includes 150 miles, and then charges \$0.75 for each mile driven over 150 miles. Plan B costs \$175 per day and includes 200 miles, and then charges \$0.50 for each mile driven over 200 miles.

(a) Find a piecewise defined function for one day with Plan A and Plan B in terms of the number of miles driven x.

(b) After how many miles will Plan B cost less than Plan A?

Solution. (a) For Plan A, we have a flat rate of \$100 for $0 \leq x \leq 150$, so $C_A(x) = 100$ on this interval. For $x > 150$, we have a rate (slope) of .75, and a point on this line is $(150, 100)$. So the equation of the line is given by

$$\begin{aligned} y &= .75(x - 150) + 100 \\ &= .75x - 112.5 + 100 \\ &= .75x - 12.5. \end{aligned}$$

So

$$C_A(x) = \begin{cases} 100, & \text{if } 0 \leq x \leq 150 \\ .75x - 12.5, & \text{if } x > 150 \end{cases}.$$

For B we do the same thing. For $0 \leq x \leq 200$, $C_B(x) = 175$. Now $(200, 175)$ is the point where the function starts increasing with a slope of .5. So for $x > 200$, we use point-slope form again:

$$\begin{aligned} y &= .5(x - 200) + 175 \\ &= .5x - 100 + 175 \\ &= .5x + 75. \end{aligned}$$

Putting this together,

$$C_B(x) = \begin{cases} 175, & \text{if } 0 \leq x \leq 200 \\ .5x + 175, & \text{if } x > 200 \end{cases}.$$

(b) If we plug in 200 to the cost function for A, we see that $C_A(200) = 137.5 < 175$. So for up to 200 miles, Plan A is still cheaper than Plan B. To see when B will become cheaper, we check when they are equal.

$$\begin{aligned} .75x - 12.5 &= .5x + 75 \\ .25x &= 87.5 \\ x &= 350. \end{aligned}$$

So when $x = 350$, the cost of each plan is equal. So for $x > 350$ miles, Plan B will be cheaper. □

Example 7. A salesman working on commission earns 10% on all sales made up to \$5000. He can then earn 15% commission on the next \$10,000 in sales, and 20% on all sales above that. Find a piecewise function C for the salesman's commission on the total sales of x dollars.

Solution. For $0 \leq x \leq 5000$, he gets a rate of .1. So on this interval $C(x) = .1x$. Then for $5000 < x \leq 10,000$, that rate changes to .15. We need a point on this line to figure out the equation. $(5000, 500)$ is a point on the line since when the sales total \$5000, he still gets 10% commission, but for the next cent, the rate increases to 15%. Using point-slope form:

$$\begin{aligned} y &= .15(x - 5000) + 500 \\ &= .15x - 750 + 500 \\ &= .15x - 250. \end{aligned}$$

Similarly, for the rate of 25%, $(10000, 1500)$ is a point on this line.

$$\begin{aligned} y &= .25(x - 10000) + 1500 \\ &= .25x - 2500 + 1500 \\ &= .25x - 1000. \end{aligned}$$

So the commission function can be written as

$$C(x) = \begin{cases} .1x, & \text{if } 0 \leq x \leq 5,000 \\ .15x - 250, & \text{if } 5,000 < x \leq 10,000 \\ .25x - 1000, & \text{if } x > 10,000 \end{cases} \qquad \square$$

Remark. The key for this lesson is to pay attention to the domain given in the problem. You also need to consider the context of the problem to determine if there is a jump in the piecewise function.